The mth Ratio Test: New Convergence Tests for Series

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The famous ratio test of d'Alembert for convergence of series depends on the limit of the simple ratio $\frac{a_{n+1}}{a_n}$ (J. d'Alembert, 1717–1783). If the limit is 1, the test fails. Most notable is its failure in situations where it is expected to succeed. For example, it often fails on series with terms containing factorials or finite products. Such terms appear in Taylor series of many functions.

The frequent failure of the ratio test motivated many mathematicians to analyze the ratio $\frac{a_{n+1}}{a_n}$ when its limit is 1. Of course, if the limit of $\frac{a_{n+1}}{a_n}$ is 1, then $\frac{a_{n+1}}{a_n} = 1 + b_n$ for some sequence b_n that converges to 0. A close look at b_n leads to several sharper tests than the ratio test, such as Kummer's, Raabe's, and Gauss's tests.

For example, the test which is due to J. L. Raabe (1801–1859) covers some series with factorial terms where the ratio test fails. Some series which are not covered by Raabe's test can be tested with the sharper test of C. F. Gauss (1777–1855). In fact, Gauss's test was devised to test the hypergeometric series with unit argument $F(\alpha, \beta; \gamma; 1)$; here

$$F(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)\beta(\beta+1)(\beta+2)\cdots(\beta+n-1)}{n!\,\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-1)} x^n.$$

In this paper we will give new ratio tests for convergence of series together with several examples of series where these new ratio tests succeed but the ordinary ratio test fails. We will prove that convergence by the ratio test implies convergence by these new tests.

Our main convergence test, the second ratio test, is stated in Theorem 1. The test depends on the two ratios $\frac{a_{2n}}{a_n}$ and $\frac{a_{2n+1}}{a_n}$. The simple nature of these ratios makes this test a simple convergence test.

The examples given in this paper will show that this test applies to a wide range of series, including series that appear to require Raabe's or Gauss's test. Covering cases that require delicate tests like Raabe's or Gauss's test is one of the strengths of this test. A second strength is its success, with few calculations, in testing series given in a typical calculus book or an advanced calculus book.

To further show the wide range of applications of these new convergence tests, we will use the second ratio test to give a new proof of Raabe's test. Then, we will conclude this paper with a new ratio comparison test that uses these new ratios.

For convenience, we list the tests of Kummer, Raabe, and Gauss [**2**].

Theorem (Kummer's Test). *If $a_n > 0$, $d_n > 0$, $\sum_{n=1}^{\infty} d_n$ diverges, and*

$$\lim_{n\to\infty}\left(\frac{1}{d_n} - \frac{a_{n+1}}{a_n}\cdot\frac{1}{d_{n+1}}\right) = h,$$

then $\sum_{n=1}^{\infty} a_n$ converges if $h > 0$ and diverges if $h < 0$.

Theorem (Raabe's Test). *If* $a_n > 0$, $\epsilon_n \to 0$, *and*

$$\frac{a_{n+1}}{a_n} = 1 - \frac{\beta}{n} + \frac{\epsilon_n}{n},$$

where β *is independent of* n, *then* $\sum_{n=1}^{\infty} a_n$ *converges if* $\beta > 1$ *and diverges if* $\beta < 1$.

Theorem (Gauss's Test). *If* $a_n > 0$, θ_n *is bounded, and* $\frac{a_{n+1}}{a_n} = 1 - \frac{\beta}{n} + \frac{\theta_n}{n^{1+\lambda}}$, $\lambda > 0$, *where* β *is independent of* n, *then* $\sum_{n=1}^{\infty} a_n$ *converges if* $\beta > 1$ *and diverges if* $\beta \leq 1$.

Our main result, the second ratio test, is given in the following Theorem.

Theorem 1 (The Second Ratio Test). *Let* $\sum_{n=1}^{\infty} a_n$ *be a positive-term series. Let*

$$L = \max\left\{\limsup_{n\to\infty} \frac{a_{2n}}{a_n}, \limsup_{n\to\infty} \frac{a_{2n+1}}{a_n}\right\}$$

and

$$l = \min\left\{\liminf_{n\to\infty} \frac{a_{2n}}{a_n}, \liminf_{n\to\infty} \frac{a_{2n+1}}{a_n}\right\}.$$

(i) *If* $L < \frac{1}{2}$, *then* $\sum_{n=1}^{\infty} a_n$ *converges.*
(ii) *If* $l > \frac{1}{2}$, *then* $\sum_{n=1}^{\infty}$ *diverges.*
(iii) *If* $l \leq \frac{1}{2} \leq L$, *then the test is inconclusive.*

Proof.

(i) Suppose $L < \frac{1}{2}$. Let r be such that $L < r < \frac{1}{2}$. Then there is an integer N such that

$$\frac{a_{2n}}{a_n} \leq r \quad \text{and} \quad \frac{a_{2n+1}}{a_n} \leq r$$

for all $n \geq N$. Now,

$$\begin{aligned}\sum_{n=N}^{\infty} a_n &= (a_N + a_{N+1} + \cdots + a_{2N-1}) + (a_{2N} + a_{2N+1} + \cdots + a_{4N-1})\\ &\quad + (a_{4N} + a_{4N+1} + \cdots + a_{8N-1}) + \cdots\\ &\quad + (a_{2^kN} + a_{2^kN+1} + \cdots + a_{2^{k+1}N-1}) + \cdots\\ &= \sum_{k=0}^{\infty} (a_{2^kN} + a_{2^kN+1} + \cdots + a_{2^{k+1}N-1}).\end{aligned}$$

Let $S_k = a_{2^kN} + a_{2^kN+1} + \cdots + a_{2^{k+1}N-1}$ for $k = 0, 1, 2, 3, \ldots$. Then, for $k \geq 1$,

$$S_k = (a_{2^kN} + a_{2^kN+1}) + (a_{2^kN+2} + a_{2^kN+3}) + \cdots + (a_{2^{k+1}N-2} + a_{2^{k+1}N-1}).$$

Since $\frac{a_{2n}}{a_n} \leq r$ and $\frac{a_{2n+1}}{a_n} \leq r$,

$$\begin{aligned}
S_k &= (a_{2^k N} + a_{2^k N+1}) + (a_{2^k N+2} + a_{2^k N+3}) + \cdots + (a_{2^{k+1} N-2} + a_{2^{k+1} N-1}) \\
&\leq 2(a_{2^{k-1} N})r + 2(a_{2^{k-1} N+1})r + \cdots + 2(a_{2^k N-1})r \\
&= 2r(a_{2^{k-1} N} + a_{2^{k-1} N+1} + \cdots + a_{2^k N-1}) = 2r S_{k-1}.
\end{aligned}$$

So, by induction on k we can show that

$$S_k \leq 2^k r^k (a_N + a_{N+1} + \cdots + a_{2N-1}) = 2^k r^k S_0$$

for $k \geq 1$. Thus,

$$\sum_{n=N}^{\infty} a_n = \sum_{k=0}^{\infty} S_k \leq \sum_{k=0}^{\infty} S_0 (2r)^k < \infty$$

since $r < \frac{1}{2}$. Therefore, $\sum_{n=\infty}^{\infty} a_n$ converges if $L < \frac{1}{2}$.

(ii) Suppose $l > \frac{1}{2}$. Let r be such that $\frac{1}{2} < r < l$. Then there is an integer N such that

$$\frac{a_{2n}}{a_n} > r \quad \text{and} \quad \frac{a_{2n+1}}{a_n} > r$$

for all $n \geq N$. Thus, $a_{2n} > r a_n$ and $a_{2n+1} > r a_n$ for all $n \geq N$. Let S_k be as above. It can be shown by induction that $S_k \geq S_0 (2r)^k$ for $k \geq 1$. Therefore, since $r > \frac{1}{2}$, we have

$$\sum_{n=N}^{\infty} a_n = \sum_{k=0}^{\infty} S_k \geq \sum_{k=0}^{\infty} S_0 (2r)^k = \infty.$$

Thus, $\sum_{n=\infty}^{\infty} a_n$ diverges if $l > \frac{1}{2}$.

(iii) The series $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{1}{n(\ln n)^p}$ converges if $p > 1$ and diverges if $p \leq 1$. But

$$\lim_{n\to\infty} \frac{a_{2n}}{a_n} = \lim_{n\to\infty} \frac{n(\ln n)^p}{2n[\ln(2n)]^p} = \frac{1}{2}.$$

This completes the proof. ∎

In many examples $\lim_{n\to\infty} \frac{a_{2n}}{a_n}$ and $\lim_{n\to\infty} \frac{a_{2n+1}}{a_n}$ exist, and in this case the second ratio test takes a simpler form. Using the notation of Theorem 1, if $L_1 = \lim_{n\to\infty} \frac{a_{2n}}{a_n}$ and $L_2 = \lim_{n\to\infty} \frac{a_{2n+1}}{a_n}$, then

$$\limsup_{n\to\infty} \frac{a_{2n}}{a_n} = \liminf_{n\to\infty} \frac{a_{2n}}{a_n} = L_1, \quad \limsup_{n\to\infty} \frac{a_{2n+1}}{a_n} = \liminf_{n\to\infty} \frac{a_{2n+1}}{a_n} = L_2.$$

Thus, we have the following corollary.

Corollary 1. *Let $\sum_{n=1}^{\infty} a_n$ be a positive-term series. Suppose*

$$\lim_{n\to\infty} \frac{a_{2n}}{a_n} \quad \textit{and} \quad \lim_{n\to\infty} \frac{a_{2n+1}}{a_n}$$

exist. Let $L_1 = \lim_{n\to\infty} \frac{a_{2n}}{a_n}$, $L_2 = \lim_{n\to\infty} \frac{a_{2n+1}}{a_n}$, $L = \max\{L_1, L_2\}$, *and* $l = \min\{L_1, L_2\}$.

(i) *If* $L < \frac{1}{2}$*, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

(ii) *If* $l > \frac{1}{2}$*, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

(iii) *If* $l \le \frac{1}{2} \le L$*, then the test is inconclusive.*

In Corollary 1, if we further assume that $\{a_n\}$ is a decreasing sequence, then $L = \lim_{n\to\infty} \frac{a_{2n}}{a_n}$ and $l = \lim_{n\to\infty} \frac{a_{2n+1}}{a_n}$, and therefore we have the following corollary.

Corollary 2. *If* $\{a_n\}$ *is a positive decreasing sequence, then* $\sum_{n=1}^{\infty} a_n$ *converges if* $\lim_{n\to\infty} \frac{a_{2n}}{a_n} < \frac{1}{2}$ *and diverges if* $\lim_{n\to\infty} \frac{a_{2n+1}}{a_n} > \frac{1}{2}$.

We would like to point out that Corollary 2 is related to Cauchy's Theorem from which it can be proved.

Theorem (Cauchy). *If* $\{a_n\}$ *is decreasing and* $\lim_{n\to\infty} a_n = 0$*, then both* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{k=0}^{\infty} 2^n a_{2^n}$ *converge or both diverge.*

To see that Corollary 2 follows from Cauchy's Theorem, notice that

$$l = \lim_{n\to\infty} \frac{a_{2n+1}}{a_n} \le \lim_{n\to\infty} \frac{a_{2n}}{a_n} = L.$$

(i) If $L < \frac{1}{2}$, then, using the ordinary ratio test on $\sum_{k=0}^{\infty} 2^n a_{2^n}$, we obtain

$$\lim_{n\to\infty} \frac{2^{n+1} a_{2^{n+1}}}{2^n a_{2^n}} = 2 \lim_{n\to\infty} \frac{a_{2(2^n)}}{a_{2^n}} = 2L < 1.$$

Therefore, $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges. Thus, $\sum_{n=0}^{\infty} a_n$ converges.

(ii) If $l > \frac{1}{2}$, then

$$\lim_{n\to\infty} \frac{2^{n+1} a_{2^{n+1}}}{2^n a_{2^n}} = 2 \lim_{n\to\infty} \frac{a_{2(2^n)}}{a_{2^n}} \ge 2l > 1.$$

Therefore, $\sum_{n=0}^{\infty} 2^n a_{2^n}$ diverges. Thus, $\sum_{n=0}^{\infty} a_n$ diverges.

Cauchy's Theorem requires that the sequence $\{a_n\}$ be a decreasing sequence. This is a strong assumption on the sequence. It leads, as you can see from the above proof, to a stronger version of Corollary 2, namely: if $\{a_n\}$ is a decreasing sequence, then $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} \frac{a_{2n}}{a_n} < \frac{1}{2}$ and diverges if $\lim_{n\to\infty} \frac{a_{2n}}{a_n} > \frac{1}{2}$.

Now we turn to the issue of the strength of these tests. To understand the strength of these tests and their connection to the ordinary ratio test, we need to look at the two relations

$$\frac{a_{2n}}{a_n} = \frac{a_{n+1}}{a_n} \cdot \frac{a_{n+2}}{a_{n+1}} \cdots \frac{a_{2n}}{a_{2n-1}}$$

and

$$\frac{a_{2n+1}}{a_n} = \frac{a_{n+1}}{a_n} \cdot \frac{a_{n+2}}{a_{n+1}} \cdots \frac{a_{2n+1}}{a_{2n}}.$$

From these relations it is easy to see that convergence by the ordinary ratio test implies $\lim_{n\to\infty} \frac{a_{2n}}{a_n} = \lim_{n\to\infty} \frac{a_{2n+1}}{a_n} = 0$, and therefore implies convergence by the second ratio tests. Also, divergence by the ratio test implies $\lim_{n\to\infty} \frac{a_{2n}}{a_n} = \lim_{n\to\infty} \frac{a_{2n+1}}{a_n} = \infty$, and therefore implies divergence by the second ratio test.

To make a clear distinction between these new tests and the ordinary ratio test, we give several examples of series on which the ordinary ratio test fails, but the second ratio test succeeds. For convenience, we list below two limit formulas that are used several times in the calculations of these examples:

$$\lim_{n\to\infty} \left[\frac{tn+\alpha}{tn}\right]^{n-a} = e^{\alpha/t}$$

and

$$\lim_{n\to\infty} \left[\frac{tn+\beta}{tn+\gamma}\right]^{n-a} = e^{(\beta-\gamma)/t}.$$

They can be obtained directly or from the well-known limit formula

$$\lim_{n\to\infty} \left[1+\frac{\alpha}{n}\right]^{n} = e^{\alpha}.$$

Example 1 (The *p*-series). Let $a_n = \frac{1}{n^p}$. Then $\frac{a_{2n}}{a_n} = \frac{1}{2^p}$ and $\frac{a_{2n+1}}{a_n} = \frac{1}{(2+\frac{1}{n})^p}$. Therefore, $\lim_{n\to\infty} \frac{a_{2n+1}}{a_n} = \lim_{n\to\infty} \frac{a_{2n}}{a_n} = \frac{1}{2^p}$. Since $\frac{1}{2^p} < \frac{1}{2}$ if $p > 1$ and $\frac{1}{2^p} > \frac{1}{2}$ if $p < 1$, this proves the well-known fact that the series $\sum_{n=0}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p < 1$.

Example 2. Let $a_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n(n+1)!}$. Then

$$\begin{aligned}\frac{a_{2n+1}}{a_n} < \frac{a_{2n}}{a_n} &= \frac{(2n+1)(2n+3)\cdots(4n-1)}{2^n(n+2)(n+3)\cdots(2n)(2n+1)} = \frac{(2n+3)(2n+5)\cdots(4n-1)}{2^n(n+2)(n+3)\cdots(2n)}\\ &= \frac{1}{2}\left(\frac{2n+3}{2n+4}\right)\left(\frac{2n+5}{2n+6}\right)\cdots\left(\frac{4n-1}{4n}\right)\\ &< \frac{1}{2}\left(\frac{4n-1}{4n}\right)^{n-1} = \frac{1}{2}\left(1-\frac{1}{4n}\right)^{n-1}.\end{aligned}$$

Since $\lim_{n\to\infty} \left(1-\frac{1}{4n}\right)^{n-1} = \frac{1}{\sqrt[4]{e}}$,

$$\limsup_{n\to\infty} \frac{a_{2n+1}}{a_n} \le \limsup_{n\to\infty} \frac{a_{2n}}{a_n} \le \frac{1}{2\sqrt[4]{e}} < \frac{1}{2}.$$

Therefore, the series $\sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n(n+1)!}$ converges.

Example 3. Let $x > 0$ and let $a_n = \frac{(n-1)!}{(1+x)(2+x)(3+x)\cdots(n+x)}$. Then

$$\frac{a_{2n+1}}{a_n} \le \frac{a_{2n}}{a_n} = \frac{(n)(n+1)(n+2)\cdots(2n-1)}{(n+1+x)(n+2+x)\cdots(2n-1+x)(2n+x)}$$

$$= \frac{n}{2n+x}\left(\frac{n+1}{n+1+x}\right)\left(\frac{n+2}{n+2+x}\right)\cdots\left(\frac{2n-1}{2n-1+x}\right)$$
$$\leq \frac{n}{2n+x}\left(\frac{2n-1}{2n-1+x}\right)^{n-1} = \frac{n}{2n+x}\left(1-\frac{x}{2n-1+x}\right)^{n-1}.$$

It is easy to see that

$$\lim_{n\to\infty} \frac{n}{2n+x}\left(1-\frac{x}{2n-1+x}\right)^{n-1} = \frac{1}{2}e^{-x/2}.$$

Thus,

$$\limsup_{n\to\infty} \frac{a_{2n+1}}{a_n} \leq \limsup_{n\to\infty} \frac{a_{2n}}{a_n} \leq \frac{1}{2}e^{-x/2} < \frac{1}{2}.$$

Therefore, the series

$$\sum_{n=1}^{\infty} \frac{(n-1)!}{(1+x)(2+x)(3+x)\cdots(n+x)}$$

converges for $x > 0$.

Example 4. Let $x > 0$ and let $a_n = \frac{1\cdot 2^p\cdot 3^p\cdots(n-1)^p}{(1+x)(2^p+x)(3^p+x)\cdots(n^p+x)}$. Then

$$\frac{a_{2n+1}}{a_n} \leq \frac{a_{2n}}{a_n} = \frac{(n^p)(n+1)^p(n+2)^p\cdots(2n-1)^p}{((n+1)^p+x)((n+2)^p+x)\cdots((2n-1)^p+x)((2n)^p+x)}$$
$$\leq \frac{n^p}{(2n)^p+x}.$$

Therefore,

$$\limsup_{n\to\infty} \frac{a_{2n+1}}{a_n} \leq \limsup_{n\to\infty} \frac{a_{2n}}{a_n} \leq \frac{1}{2^p} < \frac{1}{2} \quad \text{for } p > 1.$$

Also, it is easy to see that

$$\frac{a_{2n+1}}{a_n} \leq \frac{a_{2n}}{a_n} = \frac{(n^p)(n+1)^p(n+2)^p\cdots(2n-1)^p}{((n+1)^p+x)((n+2)^p+x)\cdots((2n-1)^p+x)((2n)^p+x)}$$
$$\leq \left(\frac{n^p}{(2n)^p+x}\right)\left(1-\frac{x}{(2n-1)^p+x}\right)^{n-1}.$$

Thus, if $p < 1$ then

$$\limsup_{n\to\infty} \frac{a_{2n+1}}{a_n} \leq \limsup_{n\to\infty} \frac{a_{2n}}{a_n} \leq \lim_{n\to\infty} \left(\frac{n^p}{(2n)^p+x}\right)\left(1-\frac{x}{(2n-1)^p+x}\right)^{n-1} = 0.$$

Therefore, if $x > 0$, the series $\sum_{n=1}^{\infty} \frac{1\cdot 2^p\cdot 3^p\cdots(n-1)^p}{(1+x)(2^p+x)(3^p+x)\cdots(n^p+x)}$ converges for all $p < 1$ or $p > 1$. From this example and Example 3 the series converges for all $x > 0$ and all p.

Examples 3 and 4 are both special cases of problem E3416, which appeared in the MONTHLY problems and solutions section [**3**].

Example 5. Let $a_n = \left[\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\, n!}\right]^p$. Then

$$\frac{a_{2n+1}}{a_n} < \frac{a_{2n}}{a_n} = \left[\frac{(2n+1)(2n+3)(2n+5)\cdots(4n-1)}{2^n(n+1)(n+2)\cdots(2n)}\right]^p$$
$$= \left[\left(1-\frac{1}{2n+1}\right)\left(1-\frac{1}{2n+4}\right)\cdots\left(1-\frac{1}{4n}\right)\right]^p.$$

Now, since $1-x \le e^{-x}$ for $0 < x < 1$,

$$\frac{a_{2n+1}}{a_n} < \frac{a_{2n}}{a_n} \le e^{-p\left(\frac{1}{2n+2}+\frac{1}{2n+4}+\cdots+\frac{1}{4n}\right)}.$$

But

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n} > \ln\left[\frac{2n+1}{n+1}\right],$$

so

$$\frac{a_{2n+1}}{a_n} < \frac{a_{2n}}{a_n} \le e^{-\frac{p}{2}\ln\left[\frac{2n+1}{n+1}\right]} = \left[\frac{2n+1}{n+1}\right]^{-p/2}.$$

Thus,

$$\limsup_{n\to\infty}\frac{a_{2n+1}}{a_n} \le \limsup_{n\to\infty}\frac{a_{2n}}{a_n} \le \lim_{n\to\infty}\left[\frac{2n+1}{n+1}\right]^{-p/2} = 2^{-p/2} < \frac{1}{2}$$

if $p > 2$. Therefore, the series $\sum_{n=0}^{\infty}\left[\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\, n!}\right]^p$ converges if $p > 2$. If $p \le 2$, this series diverges since $a_n \ge \frac{1}{4n}$.

Example 5 often appears in calculus books as an exercise on Gauss's test. (For example, see [**1**, Exercise 18, p. 403].)

Example 6 (Hypergeometric series). Let α, β, and γ be positive numbers. Let

$$a_n = \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)\beta(\beta+1)(\beta+2)\cdots(\beta+n-1)}{n!\,\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-1)}.$$

Then

$$\frac{a_{2n}}{a_n} = \frac{(\alpha+n)(\alpha+n+1)\cdots(\alpha+2n-1)(\beta+n)(\beta+n+1)\cdots(\beta+2n-1)}{(n+1)(n+2)\cdots(2n)(\gamma+n)(\gamma+n+1)\cdots(\gamma+2n-1)}$$
$$= \frac{(\alpha+n)(\beta+n)}{2n(\gamma+n)}\cdot\left[\frac{(\alpha+n+1)(\beta+n+1)}{(n+1)(\gamma+n+1)}\cdot\frac{(\alpha+n+2)(\beta+n+1)}{(n+2)(\gamma+n+1)}\right.$$
$$\left.\cdots\frac{(\alpha+2n-1)(\beta+2n-1)}{(2n-1)(\gamma+2n-1)}\right].$$

Each of the rational expressions inside the brackets is of the form $\frac{(\alpha+x)(\beta+x)}{x(\gamma+x)}$.

Let $f(x) = \frac{(\alpha+x)(\beta+x)}{x(\gamma+x)}$. Then

$$f(x) = \frac{(\alpha + x)(\beta + x)}{x(\gamma + x)} = 1 + \frac{(\alpha + \beta - \gamma)x + \alpha\beta}{x(\gamma + x)}.$$

So, if $\alpha + \beta < \gamma$, then there is some $N > 0$ such that $f(x)$ is increasing for all $x > N$.

Therefore, if $\alpha + \beta < \gamma$, then

$$\frac{a_{2n}}{a_n} \leq \frac{(\alpha + n)(\beta + n)}{2n(\gamma + n)} \cdot \left[\frac{(\alpha + 2n - 1)(\beta + 2n - 1)}{(2n - 1)(\gamma + 2n - 1)}\right]^{n-2}$$

for all $n > N$. Similarly

$$\frac{a_{2n+1}}{a_n} \leq \frac{(\alpha + n)(\beta + n)}{(2n + 1)(\gamma + n)} \cdot \left[\frac{(\alpha + 2n)(\beta + 2n)}{(2n)(\gamma + 2n)}\right]^{n-1}$$

for all $n > N$. From this it is easy to show that

$$\limsup_{n\to\infty} \frac{a_{2n}}{a_n} \leq \frac{1}{2} e^{\frac{\alpha+\beta-\gamma}{2}} < \frac{1}{2}$$

and similarly

$$\limsup_{n\to\infty} \frac{a_{2n+1}}{a_n} < \frac{1}{2}$$

if $\alpha + \beta < \gamma$. Thus, for positive numbers α, β, and γ, the hypergeometric series

$$\sum_{n=0}^{\infty} \frac{\alpha(\alpha + 1)(\alpha + 2)\cdots(\alpha + n - 1)\beta(\beta + 1)(\beta + 2)\cdots(\beta + n - 1)}{n!\,\gamma(\gamma + 1)(\gamma + 2)\cdots(\gamma + n - 1)}$$

converges if $\alpha + \beta < \gamma$.

If $\alpha + \beta \geq \gamma$, then $\frac{(\alpha+x)(\beta+x)}{x(\gamma+x)} = 1 + \frac{(\alpha+\beta-\gamma)x+\alpha\beta}{x(\gamma+x)} > 1$ for all $x > 0$. From this, one can show that

$$\frac{\alpha(\alpha + 1)(\alpha + 2)\cdots(\alpha + n - 1)\beta(\beta + 1)(\beta + 2)\cdots(\beta + n - 1)}{n!\,\gamma(\gamma + 1)(\gamma + 2)\cdots(\gamma + n - 1)} > \frac{\alpha\beta}{\gamma n}$$

for all n. Therefore, the series diverges if $\alpha + \beta \geq \gamma$.

In applying the second ratio test, a minor detail like reindexing the series can affect the limits that occur in the test. This might not be expected by someone who is accustomed to only the basic standard tests, such as the ratio test. To see this, consider the series:

$$\frac{1}{2} + 1 + \frac{1}{2^5} + \frac{1}{2^4} + \frac{1}{2^3} + \frac{1}{2^2} + \frac{1}{2^{13}} + \frac{1}{2^{12}} + \cdots + \frac{1}{2^6} + \frac{1}{2^{29}} + \frac{1}{2^{28}} + \cdots$$
$$+ \frac{1}{2^{14}} + \cdots + \frac{1}{2^{2(2^k-2)+1}} + \cdots + \frac{1}{2^{2^k-1}} + \frac{1}{2^{2^k-2}} + \cdots$$

If this series is indexed so that the first term is term number 1, then the second ratio test is inconclusive. But if the terms are numbered starting with 2, then the second ratio test determines that the series converges.

Next, we state the mth Ratio Test in Theorem 2 and we state Corollary 3 as one of its corollaries. The mth ratio test is a generalization of the second ratio test. The proofs of Theorem 2 and Corollary 3 are similar to the proofs of Theorem 1 and Corollary 2.

Theorem 2 (The *m*th Ratio Test). *Let $\{a_n\}$ be a positive sequence and let $m > 1$ be a fixed positive integer. Let $L_1 = \limsup_{n\to\infty} \frac{a_{mn}}{a_n}$, $L_2 = \limsup_{n\to\infty} \frac{a_{mn+1}}{a_n}, \ldots,$ and $L_m = \limsup_{n\to\infty} \frac{a_{mn+m-1}}{a_n}$. Let $l_1 = \liminf_{n\to\infty} \frac{a_{mn}}{a_n}$, $l_2 = \liminf_{n\to\infty} \frac{a_{mn+1}}{a_n}, \ldots,$ and $l_m = \liminf_{n\to\infty} \frac{a_{mn+m-1}}{a_n}$. Let $L = \max\{L_1, L_2, \ldots, L_m\}$ and $l = \min\{l_1, l_2, \ldots, l_m\}$.*

(i) *If $L < \frac{1}{m}$, then $\sum_{n=1}^{\infty} a_n$ converges.*

(ii) *If $l > \frac{1}{m}$, then $\sum_{n=1}^{\infty} a_n$ diverges.*

(iii) *If $l \le \frac{1}{m} \le L$, then the test is inconclusive.*

Corollary 3. *Let m be a fixed positive integer. If $\{a_n\}$ is a positive decreasing sequence, then $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} \frac{a_{mn}}{a_n} < \frac{1}{m}$ and diverges if $\lim_{n\to\infty} \frac{a_{mn+m-1}}{a_n} > \frac{1}{m}$.*

Now we use Theorem 1 to give a new proof for the first half (the convergence half) of Raabe's test. This is the nontrivial half of the proof. The second half (the divergence half) is relatively simple because the assumption in Raabe's Test on a_n implies that $a_n > \frac{M}{n}$ for large n and some constant M, so we will skip the divergence part of the proof.

Theorem (Raabe's Test). *If $a_n > 0$, $\epsilon_n \to 0$, and $\frac{a_{n+1}}{a_n} = 1 - \frac{\beta}{n} + \frac{\epsilon_n}{n}$, where β is independent of n, then $\sum_{n=1}^{\infty} a_n$ converges if $\beta > 1$ and diverges if $\beta < 1$.*

Proof. Suppose $\frac{a_{n+1}}{a_n} = 1 - \frac{\beta}{n} + \frac{\epsilon_n}{n}$, where $a_n > 0$ and $\epsilon_n \to 0$. Assume $1 < \beta$, and choose α such that $1 < \alpha < \beta$. Then there is some N such that

$$\frac{a_{n+1}}{a_n} < 1 - \frac{\alpha}{n}$$

for $n \ge N$. Then

$$\frac{a_{2n}}{a_n} = \frac{a_{n+1}}{a_n} \cdot \frac{a_{n+2}}{a_{n+1}} \cdots \frac{a_{2n}}{a_{2n-1}} < \left(1 - \frac{\alpha}{n}\right) \cdots \left(1 - \frac{\alpha}{2n-1}\right)$$

for $n \ge N$. Since $1 - x \le e^{-x}$ for $0 < x < 1$,

$$\left(1 - \frac{\alpha}{n}\right) \cdot \left(1 - \frac{\alpha}{n+1}\right) \cdots \left(1 - \frac{\alpha}{2n-1}\right) \le e^{-\left(\frac{\alpha}{n} + \frac{\alpha}{n+1} + \cdots + \frac{\alpha}{2n-1}\right)},$$

and since $\frac{\alpha}{n} + \frac{\alpha}{n+1} + \cdots + \frac{\alpha}{2n-1} > \alpha \ln\left(\frac{2n}{n}\right) = \alpha \ln 2$, we have

$$\left(1 - \frac{\alpha}{n}\right) \cdot \left(1 - \frac{\alpha}{n+1}\right) \cdots \left(1 - \frac{\alpha}{2n-1}\right) \le e^{-\alpha \ln 2} = \frac{1}{2^\alpha}.$$

Similarly,

$$\frac{a_{2n+1}}{a_n} = \frac{a_{n+1}}{a_n} \cdot \frac{a_{n+2}}{a_{n+1}} \cdots \frac{a_{2n+1}}{a_{2n}}$$

$$< \left(1-\frac{\alpha}{n}\right)\cdot\left(1-\frac{\alpha}{n+1}\right)\cdots\left(1-\frac{\alpha}{2n}\right)$$
$$\leq e^{-\left(\frac{\alpha}{n}+\frac{\alpha}{n+1}+\cdots+\frac{\alpha}{2n}\right)}$$
$$\leq e^{-\alpha \ln\frac{2n+1}{n}} \leq e^{-\alpha\ln 2} = \frac{1}{2^\alpha}.$$

Thus

$$\limsup_{n\to\infty}\frac{a_{2n+1}}{a_n} \leq \frac{1}{2^\alpha} < \frac{1}{2} \quad \text{and} \quad \limsup_{n\to\infty}\frac{a_{2n}}{a_n} \leq \frac{1}{2^\alpha} < \frac{1}{2}.$$

Therefore, by Theorem 1, $\sum_{n=1}^{\infty} a_n$ converges. This completes the proof. ■

Our final result is the second ratio comparison test. This test uses the ratios appearing in the second ratio test.

As we know, the proof of the ordinary ratio comparison test depends on the direct comparison test. This is also the case with the second ratio comparison test, as we will see in the proof of Theorem 3 below.

We would like to point out that the ratio comparison test is rarely used in computations, but it has very important theoretical uses. We expect this to be the case with the second ratio comparison test as well.

Theorem 3 (Ratio Comparison Test). *Suppose $\{a_n\}$ and $\{b_n\}$ are positive sequences. If $\frac{a_{2n}}{a_n} \leq \frac{b_{2n}}{b_n}$ and $\frac{a_{2n+1}}{a_n} \leq \frac{b_{2n+1}}{b_n}$ for large n, then $\sum_{n=1}^{\infty} a_n$ converges if $\sum_{n=1}^{\infty} b_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges if $\sum_{n=1}^{\infty} a_n$ diverges.*

Proof. Let N be such that

$$\frac{a_{2n}}{a_n} \leq \frac{b_{2n}}{b_n} \quad \text{and} \quad \frac{a_{2n+1}}{a_n} \leq \frac{b_{2n+1}}{b_n} \quad \text{for } n \geq N. \tag{1}$$

Let $M = \max\left\{\frac{a_{N+m}}{b_{N+m}} : m = 0, 1, \ldots, N-1\right\}$. Then we have

$$\frac{a_n}{b_n} \leq M \quad \text{for } N \leq n \leq 2N-1. \tag{2}$$

Now, we show that (1) and (2) together imply

$$\frac{a_n}{b_n} \leq M \quad \text{for } n \geq N. \tag{3}$$

If (3) is not true, let m be the least integer for which $\frac{a_m}{b_m} > M$. Now write m as $2j$ or $2j+1$, depending on whether m is even or odd, and apply (1) to get a smaller integer than m for which (3) is not true. This is a contradiction.

Now, from (3) and the ordinary comparison test, the conclusion of the theorem follows. This completes the proof. ■

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REFERENCES

1. T. Apostol, *Calculus*, Volume I, John Wiley, 1967.
2. E. Rainville, *Infinite Series*, Macmillan, New York, 1967.
3. A. Zeifman, Problem E3416, this MONTHLY 98 (1991) 54.

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A *Simpler* Proof of a Well-Known Fact

Inspired by Xinyun Zhu's "Simple Proof of a Well-Known Fact" on page 416 of the May 2007 MONTHLY, we give an even simpler proof:

Theorem. *For a positive integer N that is not a perfect square, $\sqrt{N}$ is irrational.*

Proof. Suppose that $\sqrt{N} = a/b$ where a and b are positive integers with no common factors. Then

$$\sqrt{N} = \frac{a}{b} = \frac{Nb}{a}.$$

If two fractions are equal, with the first in lowest terms, then the numerator and denominator of the second must be a common integer multiple (say c) of the numerator and denominator of the first. Therefore, $a = bc$, so that $a/b = c$, and hence $\sqrt{N}$ is an integer, so N is a perfect square. ■

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