Math 476 - Abstract Algebra - Worksheet on Binary Operations

Binary Operations

Definition: A *binary operation* on a set S is a function that assigns to each **ordered** pair of elements of S a uniquely determined element of S. We will use the following notation

$$\begin{aligned} * : S \times S &\rightarrow S \\ a * b &:= *(a, b) \end{aligned}$$

The set S is said *closed* under the operation $*$. We call the data of a set S together with a binary operation $*$, written $(S, *)$, *magma*.

Group activity: Read and discuss the definition of binary operation. In case you do not understand some concepts feel free to ask. Then answer the following questions.

(a) Give an example of magma and use the above notation to describe it.

(b) Can you give an example of a set S and operation $*$ such that S is not closed under $*$?
(Notice that in this case $(S, *)$ is **NOT** a magma)

(c) Discuss in your group the meaning of the word "ordered" in the above definition. Give an example of binary operation such that, in general, $a * b \neq b * a$.

(d) How would you call a binary operation for which $a * b = b * a$ for all a and b in S? Give an example.

(e) Choose a member of your group to share your work with the class.

Cayley's Tables

When we have finite magmas $(S, *)$, that is to say when the set S is finite, it is convenient to represent the operation through the so called **Cayley table**. We will learn how to construct/interpret Cayley tables through examples.
Recall that the the set

$$\mathbb{Z}_n := \{0, 1, 2, \ldots, n-1\}$$

is the set of integers mod n. On this set there are usually defined two operations, $+$ and $\cdot$, called sum and multiplication mod n. You can think this operations as adding or multiplying integers with a given remainder in the division by n. For example

$$3 + 2 \equiv 1 \mod 4$$

means "if you add an integer whose remainder, in the division, by 4, is 3 to an integer whose remainder, in the division by 4, is 2, you get a number whose remainder in the division by 4 is 1". I hope you appreciate the brevity of the above expression.

Group activity: Below you have the Cayley table for the magma $(\mathbb{Z}_5, +)$, keeping in mind that the operation goes clockwise, discuss in your group the meaning of it.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

In the space below, write the Cayley tables for $(\mathbb{Z}_5, \cdot)$, $(\mathbb{Z}_6, \cdot)$, $(\mathbb{Z}_2, +)$, and $(\mathbb{Z}_2, \cdot)$, then discuss in your group if you notice some interesting pattern.

Let $S : \{a, b, c, d\}$ and consider the magma $(S, *)$ defined by the following Cayley table

*	a	b	c	d
a	c	b	d	b
b	a	d	c	c
c	d	b	a	a
d	c	c	a	b

Note that this is a magma because the binary operation is closed.
Perform the following computations in $(S, *)$.

(a) $c * b =$

(b) $b * c =$

(c) $(a * b) * c =$

(d) $a * (b * c) =$

(e) $a * b * c =$

Do you notice something unusual?

Solve the following equations for the variable x in $(S, *)$. In case you have difficulties in your group, feel free to ask.

(a) $a * x = b$

(b) $x * a = b$

(c) $x * x = c$

(d) $(x * x) * x = x * (x * x)$

Properties of Binary Operations

Definition: We say that a binary operation $*$ on a set G is *associative* if, for all a, b and c in G, we have

$$(a * b) * c = a * (b * c)$$

If in a magma $(G, *)$, the binary operation $*$ is associative, we say that $(G, *)$ is a *semigroup*.

Group Activity

(a) Write two different examples of **associative** binary operations (you should know already some associative operations in some sets of numbers or matrices), where, instead of a, b, and c, you write specific elements.

(b) Let $G = \{0, 1, 2\}$. Say whether the binary operation defined by the following Cayley table is associative. Justify your answer.

$*$	0	1	2
0	1	1	2
1	0	2	2
2	0	1	0

(c) Let $G = \{a, b\}$. Determine as many different associative binary operations as you can. Write the corresponding Cayley tables in order to define the operations.

Definition: Let G be a set and $*$ an associative binary operation on G (that is to say $(G, *)$ is a semigroup). We say that $(G, *)$ has an *identity* if there exists an element $e \in G$ (called identity) such that:

$$e * a = a * e = a$$

for all $a \in G$. A semigroup $(G, *)$ with an identity is called *monoid*.

Group Activity

(a) Discuss the definition of identity in your group. Try to list all the identities (with the related operations) you have seen so far.

(b) Show that the set of natural numbers $\mathbb{N} := \{1, 2, 3 \ldots\}$ equipped with the multiplication is a monoid. If you have difficulties feel free to ask.

(c) Prove that, in a monoid $(G, *)$, the identity is unique. Start with assuming that there are two identities e and e', your goal is to show that $e = e'$ only by using properties of identity.

(d) Let $G = \{0, 1, 2, 3\}$ equipped with the operation $*$ described in the Caley table below. Is there an identity element? If yes, what is this element? Justify your answer.

$*$	0	1	2	3
0	2	1	0	3
1	1	1	1	1
2	0	1	2	3
3	3	1	3	1

(e) How do you determine an identity from a Cayley table?

Definition: Let $(G, *)$ be a monoid (that is to say the binary operation $*$ is associative and there is an identity e). Let $x \in G$, we call *inverse of* x, written x^{-1}, an element of G such that

$$x * x^{-1} = x^{-1} * x = e.$$

Group Activity

(a) Let $\mathbb{Z} := \{\cdots - 2, -1, 0, 1, 2 \cdots\}$ be the set of integers, and consider the operation sum $+$. Is $(\mathbb{Z}, +)$ a monoid? In case it is, what is the identity? If $x = 17$ can you write x^{-1}? Is there any element of $\mathbb{Z}$ that does not have an inverse?

(b) Answer the same questions of part (a), but now consider $(\mathbb{R}, \cdot)$.

(c) Answer the same questions of part (a), but now consider $(\mathbb{Z}_6, \cdot)$. You may write the Cayley table in order to help you.

(d) Let $\mathrm{GL}(2, \mathbb{R})$ be the set of 2×2 matrices, with coefficients in $\mathbb{R}$, whose determinant is not zero, and consider as an operation the product of matrices. Is $(\mathrm{GL}(2, \mathbb{R}), \cdot)$ a monoid? Justify your answer (you may state known results from linear algebra). Let $x = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$. What is x^{-1}?