Notes Math 127	
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The intent of these notes is to summarize basic definitions and to give some examples that have been done in class. Students are recommended to read the textbook for further examples and facts.

Exponential and logarithmic equations.

1 Exponential equations

An exponential equation is an equation in which a variable occurs in the exponent. For example

$$5^x = 25$$

In this case it is not difficult to see that the solution is $x = 2$. Unfortunately some equations are not so easy to solve. We see two strategies that may be useful.

Strategy I: Write the equation in the form of two powers, with the same base, one on each side of "=":

$$a^M = a^N$$

then we can write an equality involving only the exponents:

$$M = N$$

1. **Example:** Solve $4^x = 8$ for x.

   Sol: We write each side as a power of 2.

$$\begin{aligned} (2^2)^x &= 2^3 \\ 2^{2x} &= 2^3 \\ 2x &= 3 \\ x &= \boxed{\frac{3}{2}} \end{aligned}$$

2. **Example:** Solve $5^{2x-1} = \dfrac{1}{\sqrt{5}}$

   Sol: First we write each side as a power of 5:

$$5^{2x-1} = 5^{-\frac{1}{2}}$$

then we solve the equation involving only the exponents

$$\begin{aligned} 2x - 1 &= -\frac{1}{2} \\ 2x &= 1 - \frac{1}{2} \\ 2x &= \frac{1}{2} \\ x &= \boxed{\frac{1}{4}} \end{aligned}$$

Strategy II If the first strategy does not work, take the <u>same</u> logarithm on both sides of the equation.

1. **Example:** Solve the following equation and round your answer to the second decimal place

$$2^x = 7$$

Sol. Let us take the natural logarithm on both sides and then apply the logarithmic rules

$$\begin{aligned} \ln 2^x &= \ln 7 \\ x \ln 2 &= \ln 7 \\ x &= \frac{\ln 7}{\ln 2} \approx \boxed{2.81} \end{aligned}$$

2. **Example:** Solve the following equation and round to the forth decimal place

$$\left(\frac{1}{2}\right)^{3x-2} = 15$$

Sol. Now we take the log base 10

$$\begin{aligned} \log\left(\frac{1}{2}\right)^{3x-2} &= \log 15 \\ (3x-2)\log 2^{-1} &= \log 15 \\ -(3x-2)\log 2 &= \log 15 \\ 3x - 2 &= \frac{\log 15}{-\log 2} \\ 3x &= -\frac{\log 15}{\log 2} + 2 \\ x &= \frac{2 - \frac{\log 15}{\log 2}}{3} \approx \boxed{-0.6356} \end{aligned}$$

2 Logarithmic equations

A logarithmic equation is an equation in which a variable occurs in the argument of a logarithm. For example

$$\ln x = 1.$$

Here the solution is $x = e$.

Remark: Since we are dealing with logarithms, **we have to check** if the solutions we find are in the domain of all the logarithmic expressions.

Let us see two strategies (that are basically the same) that we can use in order to solve this kind of problems.
Strategy I Write the equation in the form:

$$\log_a M = k$$

so we can write the equation in the exponential form:

$$M = a^k$$

1. **Example:** Solve the following equation and round the answer to the second decimal place

$$\ln(x-2) = 1$$

Solution: We must have $x - 2 > 0$, that is to say $\boxed{x > 2}$. The base is e, so we can write

$$\begin{aligned} x - 2 &= e^1 \\ x &= e + 2 \approx \boxed{4.72} \end{aligned}$$

The solution is acceptable because $4.72 > 2$, as we required at the beginning.

2. **Example:** Solve $\log(4x + 1) = -1$.

Solution: We must have $4x + 1 > 0$ that is to say $\boxed{x > -\frac{1}{4} = -0.25}$. The base is 10, so we can write

$$\begin{aligned} 4x + 1 &= 10^{-1} \\ 4x &= \frac{1}{10} - 1 \\ x &= \frac{-\frac{9}{10}}{4} = -\frac{9}{40} = \boxed{-0.225} \end{aligned}$$

The solution is acceptable because $-0.225 > -0.25$ (notice that we are comparing negative numbers).

3. **Example:** Solve $\log_2(x-1) + \log_2(x-2) = 1$.

   Solution: We must have $x - 1 > 0$ $\boxed{\text{and}}$ $x - 2 > 0$ that is to say $x > 1$ $\boxed{\text{and}}$ $x > 2$. This means $\boxed{x \geq 2}$.

   We now solve the equation, as a first step we condense the right side:

$$\begin{aligned} \log_2[(x-1)(x-2)] &= 1 \\ (x-1)(x-2) &= 2^1 \\ x^2 - 3x + 2 &= 2 \\ x^2 - 3x &= 0 \\ x(x-3) &= 0 \end{aligned}$$

   We get $x = 0$ or $x = 3$, but 0 is not greater than 2 as requested above. Therefore, the only acceptable solution is $\boxed{x = 3}$.

Strategy II We try to write the given equation in the form

$$\log_a M = \log_a N$$

so we can write

$$M = N$$

notice that the base of the logarithm <u>must be the same</u>.

1. **Example:** Solve $\log(x+7) - \log 3 = \log(7x+1)$

   Solution: First of all we must have $x + 7 > 0$ and $7x + 1 > 0$, in other words $x > -7$ and $x > -\frac{1}{7}$. Therefore we must have $x > -\frac{1}{7}$.

   We have (first we apply the quotient rule on the right side)

$$\begin{aligned} \log \frac{x+7}{3} &= \log(7x+1) \\ \frac{x+7}{3} &= 7x+1 \\ x+7 &= 21x+3 \\ 20x &= 4 \\ x &= \boxed{\frac{1}{5}} \end{aligned}$$

   which is <u>acceptable</u> because $\frac{1}{5} > -\frac{1}{7}$.

2. **Example:** Solve $2\log x - \log 7 = \log 112$.

   Solution: We must have $\boxed{x > 0}$.

   Now we solve. First of all we condense the right side

$$\begin{aligned} \log x^2 - \log 7 &= \log 112 \\ \log \frac{x^2}{7} &= \log 112 \\ \frac{x^2}{7} &= 112 \\ x^2 &= 784 \\ x &= \pm 28 \end{aligned}$$

   The only acceptable solution (that must be > 0 as stated above) is $\boxed{x = 28}$.