Packing Set Partitions

Adam M. Goyt
Department of Mathematics
Minnesota State University Moorhead
Moorhead, MN 56563
goytadam@mnstate.edu
www.mnstate.edu/goytadam

July 10, 2009

Outline

1. Introduction
2. Results for the Restricted Definition
3. Results for the Traditional Definition
4. Open Problems

Set Partition Definition

Definition

A partition π of a set S, written $\pi \vdash S$, is a family of disjoint nonempty subsets $B_i \subseteq S$, called blocks, such that $\uplus B_i = S$.

Set Partition Definition

Definition

A partition π of a set S, written $\pi \vdash S$, is a family of disjoint nonempty subsets $B_i \subseteq S$, called blocks, such that $\uplus B_i = S$.

We write

$$\pi = B_1/B_2/\ldots/B_k,$$

where

$$\min B_1 < \min B_2 < \cdots < \min B_k,$$

Set Partition Definition

Definition

A partition π of a set S, written $\pi \vdash S$, is a family of disjoint nonempty subsets $B_i \subseteq S$, called blocks, such that $\uplus B_i = S$.

We write

$$\pi = B_1/B_2/\dots/B_k,$$

where

$$\min B_1 < \min B_2 < \cdots < \min B_k,$$

Let

$$\Pi_n = \{\pi : \pi \vdash [n] = \{1, 2, \dots, n\}\}, \text{ and } \Pi = \bigcup_n \Pi_n.$$

Set Partition Definition

Definition

A partition π of a set S, written $\pi \vdash S$, is a family of disjoint nonempty subsets $B_i \subseteq S$, called blocks, such that $\uplus B_i = S$.

We write

$$\pi = B_1/B_2/\dots/B_k,$$

where

$$\min B_1 < \min B_2 < \cdots < \min B_k,$$

Let

$$\Pi_n = \{\pi : \pi \vdash [n] = \{1, 2, \dots, n\}\}, \text{ and } \Pi = \bigcup_n \Pi_n.$$

Example

$137/25/46 \vdash [7]$

Restricted Growth Function Definition

Definition

A restricted growth function, RGF, is a word, $r = a_1 a_2 \dots a_n \in [k]^n$*, such that* $a_1 = 1$ *and for* $i \geq 2$ *we have that* $a_i \leq 1 + \max_{j \leq i} a_j$*.*

We say that the length of $r = a_1 a_2 \dots a_n$*, denoted* $\ell(r)$*, is* n*.*

Restricted Growth Function Definition

Definition

A restricted growth function, RGF, is a word, $r = a_1 a_2 \ldots a_n \in [k]^n$*, such that* $a_1 = 1$ *and for* $i \geq 2$ *we have that* $a_i \leq 1 + \max_{j \leq i} a_j$*.*

We say that the length of $r = a_1 a_2 \ldots a_n$*, denoted* $\ell(r)$*, is* n*.*

Example

1121323412 is a RGF,

Restricted Growth Function Definition

Definition

A restricted growth function, RGF, is a word, $r = a_1 a_2 \ldots a_n \in [k]^n$*, such that* $a_1 = 1$ *and for* $i \geq 2$ *we have that* $a_i \leq 1 + \max_{j \leq i} a_j$*.*

We say that the length of $r = a_1 a_2 \ldots a_n$*, denoted* $\ell(r)$*, is* n*.*

Example

1121323412 is a RGF, but 1132413 is not.

Partitions and RGFs

We let

$$R_n = \{r : r \text{ is an RGF with } \ell(r) = n\}.$$

There is a well-known bijection $\phi : \Pi_n \to R_n$. Let $\pi = B_1/B_2/\ldots/B_k \in \Pi_n$ then

$$\pi \mapsto a_1 a_2 \ldots a_n,$$

where $a_i = j$ if $i \in B_j$.

Examples

Example

$137/25/46 \vdash [7]$.

Examples

Example

$137/25/46 \vdash [7]$.

$\phi(137/25/46) = 1213231$ *is its corresponding restricted growth function.*

Examples

Example

$137/25/46 \vdash [7]$.
$\phi(137/25/46) = 1213231$ *is its corresponding restricted growth function.*

The canonical ordering of the blocks forces ϕ to produce an RGF.

Examples

Example

$137/25/46 \vdash [7]$.
$\phi(137/25/46) = 1213231$ *is its corresponding restricted growth function.*

The canonical ordering of the blocks forces ϕ to produce an RGF. The inverse is easy to construct as well.

Examples

Example

$137/25/46 \vdash [7]$.

$\phi(137/25/46) = 1213231$ *is its corresponding restricted growth function.*

The canonical ordering of the blocks forces ϕ to produce an RGF.

The inverse is easy to construct as well.

From here on out we will work only with RGFs, but we will call them partitions.

Saintlyhood

Definition

Given any word $w \in [k]^n$ we may canonize w by replacing all occurences of the first letter of w by 1, all occurences of the next occuring letter by 2, etc.

Saintlyhood

Definition

Given any word $w \in [k]^n$ we may canonize w by replacing all occurences of the first letter of w by 1, all occurences of the next occuring letter by 2, etc.

Example

The canonized form of 47411477 is 12133122.

Pattern Containmnent

Definition

Let σ *be a partition with* $\ell(\sigma) = n$ *and* π *be a partition with* $\ell(\pi) = k$*. We say that* σ *contains* π *if*

Pattern Containmnent

Definition

Let σ be a partition with $\ell(\sigma) = n$ and π be a partition with $\ell(\pi) = k$. We say that σ contains π if

1. *Traditional:*
   there is a subsequence of σ of length k whose canonized form is order isomorphic to π.

Pattern Containmnent

Definition

Let σ be a partition with $\ell(\sigma) = n$ and π be a partition with $\ell(\pi) = k$. We say that σ contains π if

1. *Traditional:*
   there is a subsequence of σ of length k whose canonized form is order isomorphic to π.
2. *Restricted:*
   there is a subsequence of σ of length k that is order isomoriphic to π.

Examples of Containment

Example

Let $\sigma = 1213231$ *and* $\pi = 121$.

Examples of Containment

Example

Let $\sigma = 1213231$ *and* $\pi = 121$.

Then 212*, as in* 1213231 *is a copy of 121 in the traditional sense, but not in the restricted sense.*

Examples of Containment

Example

Let $\sigma = 1213231$ *and* $\pi = 121$.

Then 212*, as in* 1213231 *is a copy of 121 in the traditional sense, but not in the restricted sense.*

And 131*, as in* 1213231*, is a copy of 121 in both the traditional and restricted sense.*

Examples of Containment

Example

Let $\sigma = 1213231$ *and* $\pi = 121$.

Then 212*, as in* 1213231 *is a copy of 121 in the traditional sense, but not in the restricted sense.*

And 131*, as in* 1213231*, is a copy of 121 in both the traditional and restricted sense.*

The traditional pattern containment defintion allows for the blocks of the copy to be in a different order than the blocks in the pattern. The restricted definition does not.

Packing Definitions

Definition

Let $E \subseteq \Pi_m$ and $\sigma \in \Pi$ and $\nu(E, \sigma)$ be the number of copies of partitions from E in σ then define:

Packing Definitions

Definition

Let $E \subseteq \Pi_m$ and $\sigma \in \Pi$ and $\nu(E,\sigma)$ be the number of copies of partitions from E in σ then define:

$$\mu(E,k,n) = \max\{\nu(E,\sigma) : \sigma \vdash [n] \text{ and } \sigma \text{ has at most } k \text{ blocks}\}$$

Packing Definitions

Definition

Let $E \subseteq \Pi_m$ *and* $\sigma \in \Pi$ *and* $\nu(E,\sigma)$ *be the number of copies of partitions from* E *in* σ *then define:*

$$\begin{aligned}\mu(E,k,n) &= \max\{\nu(E,\sigma) : \sigma \vdash [n] \text{ and } \sigma \text{ has at most } k \text{ blocks}\}\\ d(E,\sigma) &= \frac{\nu(E,\sigma)}{\binom{n}{m}}\end{aligned}$$

Packing Definitions

Definition

Let $E \subseteq \Pi_m$ and $\sigma \in \Pi$ and $\nu(E,\sigma)$ be the number of copies of partitions from E in σ then define:

$$\begin{aligned}
\mu(E,k,n) &= \max\{\nu(E,\sigma) : \sigma \vdash [n] \textit{ and } \sigma \textit{ has at most } k \textit{ blocks}\} \\
d(E,\sigma) &= \frac{\nu(E,\sigma)}{\binom{n}{m}} \\
d(E,k,n) &= \frac{\mu(E,k,n)}{\binom{n}{m}}
\end{aligned}$$

Density

Theorem

For $n \geq m$ *we have that* $d(E, k, n-1) \geq d(E, k, n)$ *and* $d(E, k, n) \geq d(E, k-1, n)$. □

Density

Theorem

For $n \geq m$ we have that $d(E,k,n-1) \geq d(E,k,n)$ and $d(E,k,n) \geq d(E,k-1,n)$. □

Definition

Let the packing density of the set $E \subseteq \Pi_m$ be

$$\delta(E) = \lim_{n\to\infty} d(E,n,n) = \lim_{n\to\infty} \lim_{k\to\infty} d(E,k,n).$$

Outline

1. Introduction
2. Results for the Restricted Definition
3. Results for the Traditional Definition
4. Open Problems

Restricted Results

- The results in this section can be found in ”Packing patterns into words” by Burstein, Hästö and Mansour [2].

Restricted Results

- The results in this section can be found in ”Packing patterns into words” by Burstein, Hästö and Mansour [2].
- The restricted definition for containment is the same as the definition for pattern containment in words.

Restricted Results

- The results in this section can be found in ”Packing patterns into words” by Burstein, Hästö and Mansour [2].
- The restricted definition for containment is the same as the definition for pattern containment in words.
- A layered partition is one of the form $11\ldots122\ldots233\ldots3\ldots kk\ldots k$. (Called nondecreasing words)

Restricted Results

- The results in this section can be found in ”Packing patterns into words” by Burstein, Hästö and Mansour [2].
- The restricted definition for containment is the same as the definition for pattern containment in words.
- A layered partition is one of the form $11\dots122\dots233\dots3\dots kk\dots k$. (Called nondecreasing words)
- If E consists entirely of layered partitions then there exists a layered σ so that $\nu(E,\sigma) = \mu(E,n,n)$.

Restricted Results

- The results in this section can be found in ”Packing patterns into words” by Burstein, Hästö and Mansour [2].
- The restricted definition for containment is the same as the definition for pattern containment in words.
- A layered partition is one of the form $11\ldots122\ldots233\ldots3\ldots kk\ldots k$. (Called nondecreasing words)
- If E consists entirely of layered partitions then there exists a layered σ so that $\nu(E,\sigma) = \mu(E,n,n)$.
- All of the results of Price [3] and Albert et. al. [1] about layered permutations apply.

Restricted Results

- The results in this section can be found in ”Packing patterns into words” by Burstein, Hästö and Mansour [2].
- The restricted definition for containment is the same as the definition for pattern containment in words.
- A layered partition is one of the form $11\dots 122\dots 233\dots 3\dots kk\dots k$. (Called nondecreasing words)
- If E consists entirely of layered partitions then there exists a layered σ so that $\nu(E,\sigma)=\mu(E,n,n)$.
- All of the results of Price [3] and Albert et. al. [1] about layered permutations apply.
- $\delta(121)=\frac{2\sqrt{3}-3}{2}\approx 0.2321$

Outline

1. Introduction
2. Results for the Restricted Definition
3. Results for the Traditional Definition
4. Open Problems

The Layered Case

- Let σ be a partition, then the size of block *i* is the number of times *i* occurs in σ.

The Layered Case

- Let σ be a partition, then the size of block *i* is the number of times *i* occurs in σ.
- A pattern π will be called monotone layered if its block sizes are weakly increasing or weakly decreasing.

The Layered Case

- Let σ be a partition, then the size of block *i* is the number of times *i* occurs in σ.
- A pattern π will be called monotone layered if its block sizes are weakly increasing or weakly decreasing.
- If *E* consists entirely of monotone layered partitions then there exists some σ, which is also monotone layered, such that $\nu(E, \sigma) = \mu(E, n, n)$.

The Layered Case

- Let σ be a partition, then the size of block *i* is the number of times *i* occurs in σ.
- A pattern π will be called monotone layered if its block sizes are weakly increasing or weakly decreasing.
- If *E* consists entirely of monotone layered partitions then there exists some σ, which is also monotone layered, such that $\nu(E, \sigma) = \mu(E, n, n)$.
- Thus, all results for layered permutations that are monotone apply.

The Layered Case

- Let σ be a partition, then the size of block *i* is the number of times *i* occurs in σ.
- A pattern π will be called monotone layered if its block sizes are weakly increasing or weakly decreasing.
- If *E* consists entirely of monotone layered partitions then there exists some σ, which is also monotone layered, such that $\nu(E, \sigma) = \mu(E, n, n)$.
- Thus, all results for layered permutations that are monotone apply.
- Question: Can we get a similar result for any set of layered partitions?

Monotone Layered Theorem

Definition

The block structure of a partition, π, is the multiset of block sizes of π.

Monotone Layered Theorem

Definition

The *block structure* *of a partition, π, is the multiset of block sizes of π.*

Theorem

Given a monotone increasing layered pattern $\pi = 1 \ldots 12 \ldots 2 \ldots k \ldots k$ then among all partitions with the same block structure as σ, the layered monotone increasing one is a maximizer.

Proof

Proof: Consider $\hat{\sigma}$, the monotone increasing layered partition with the same block structure as σ. We must show that $\nu(\pi,\sigma) \leq \nu(\pi,\hat{\sigma})$.

Proof

Proof: Consider $\hat{\sigma}$, the monotone increasing layered partition with the same block structure as σ. We must show that $\nu(\pi, \sigma) \leq \nu(\pi, \hat{\sigma})$. The last letter of σ was the last occurence of some j in the partition. Let $\hat{j}$ be the letter corresponding to j in $\hat{\sigma}$.

Proof

Proof: Consider $\hat{\sigma}$, the monotone increasing layered partition with the same block structure as σ. We must show that $\nu(\pi, \sigma) \leq \nu(\pi, \hat{\sigma})$.
The last letter of σ was the last occurence of some *j* in the partition.
Let $\hat{j}$ be the letter corresponding to *j* in $\hat{\sigma}$.
By induction on the length of σ the number of copies of π in σ that do not involve this *j* is no more than the number of copies in $\hat{\sigma}$ that do not involve $\hat{j}$.

Proof

Proof: Consider $\hat{\sigma}$, the monotone increasing layered partition with the same block structure as σ. We must show that $\nu(\pi, \sigma) \leq \nu(\pi, \hat{\sigma})$.
The last letter of σ was the last occurence of some *j* in the partition.
Let $\hat{j}$ be the letter corresponding to *j* in $\hat{\sigma}$.
By induction on the length of σ the number of copies of π in σ that do not involve this *j* is no more than the number of copies in $\hat{\sigma}$ that do not involve $\hat{j}$.
Now consider the number of copies of π in σ that did involve this *j*. In any such copy the *j*s must correspond to the *k*s of π. By inducting on *k* we can say that number of copies has not decreased in this case as well. $\square$

121

Conjecture

The partition of $[n]$ *that maximizes the number of copies of* 121 *is* $\underbrace{121212\ldots}_{n}$

121

Conjecture

The partition of $[n]$ *that maximizes the number of copies of* 121 *is* $\underbrace{121212\ldots}_{n}$

Example

$$\lim_{n\to\infty} d(121, \underbrace{121212\ldots}_{n}) = \frac{1}{4}.$$

121

Conjecture

The partition of $[n]$ *that maximizes the number of copies of* 121 *is* $\underbrace{121212\ldots}_{n}$

Example

$$\lim_{n\to\infty} d(121, \underbrace{121212\ldots}_{n}) = \frac{1}{4}.$$

Theorem

$\delta(121) \leq \frac{1}{2}$. □

Outline

1. Introduction
2. Results for the Restricted Definition
3. Results for the Traditional Definition
4. Open Problems

HELP! I'm begging!

- Can we solve the layered problem?

HELP! I’m begging!

- Can we solve the layered problem?
- Can it be shown that 12121 . . . is the best possible for packing 121?

HELP! I’m begging!

- Can we solve the layered problem?
- Can it be shown that 12121 . . . is the best possible for packing 121?
- Can we at least get a better upper bound for 121?

HELP! I’m begging!

- Can we solve the layered problem?
- Can it be shown that 12121 . . . is the best possible for packing 121?
- Can we at least get a better upper bound for 121?
- Anything that might translate from the permutation case to the set partition case.

HELP! I’m begging!

- Can we solve the layered problem?
- Can it be shown that 12121 . . . is the best possible for packing 121?
- Can we at least get a better upper bound for 121?
- Anything that might translate from the permutation case to the set partition case.
- Feel free to get started right now! ☺

THANK YOU

- M. H. Albert, M. D. Atkinson, C. C. Handley, D. A. Holton, W. Stromquist, On packing densities of permutations, Electron. J. Combin. 9 (1) (2002) Research Paper 5, 20 pp. (electronic).
- A. Burstein, P. Hästö, T. Mansour, Packing patterns into words, Electron. J. Combin. 9 (2) (2002/03) Research paper 20, 13 pp. (electronic), permutation patterns (Otago, 2003).
- A. Price, Packing densities of layered patterns, Ph.D. thesis, University of Pennsylvania, Philadelphia, PA, 1997.