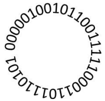
The Magic of De Bruijn Sequences



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Outline

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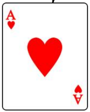

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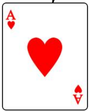

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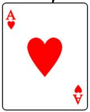

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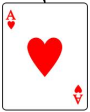

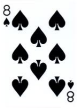

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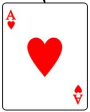

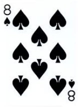

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- The de Bruijn sequence above has window length 5, and so has $2^5 = 32$ digits. Our deck consists of the 32 cards Ace through 8 of each suit.

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- The de Bruijn sequence above has window length 5, and so has $2^5 = 32$ digits. Our deck consists of the 32 cards Ace through 8 of each suit.
- Each window consists of 5 bits.
- The first 2 bits tell the suit of the card.

00	01	10	11

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- Each window consists of 5 bits.
- The final 3 bits tell the rank of the card.

000	001	010	011	100	101	110	111
8	A	2	3	4	5	6	7

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- If so how many de Bruijn sequences are there with window k?
- How do we find these de Bruijn sequences?
- Finally, are there any uses for de Bruijn sequences other than just to amaze your friends?

Existence of de Bruijn Sequences?

The answer to this question is yes. We will need a little graph theory to explain.

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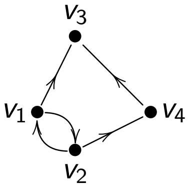

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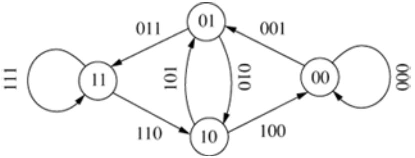

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Example

00 00 01 10 01 11 11 10 00
collapses to
00010111

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Definition

A path through a graph that traverses every edge exactly once and ends where it began is called an Eulerian circuit.

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Question

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Thus, the de Bruijn graph always has an Eulerian circuit, and hence de Bruijn sequences exist for any window length k.

How many de Bruijn Sequences are there?

We need to be able to count the number of Eulerian circuits in a directed graph.

Theorem (de Bruijn, 1946)

There are

$$2^{2^{k-1}-k}$$

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For $k = 5$ there are 2048 de Bruijn sequences.

History of de Bruijn sequences

- The existence question was first proposed by A de Rivière in the french problem journal *l'Intermédiaire des Mathématiciens* in 1894.
- The first solution was published the same year by C Flye de Sainte-Marie [5] using graph theory.
- Many others contributed between 1894 and 1946. De Bruijn later acknowledged Sainte-Marie's proof being the first, but his name stuck.

How do we find de Bruijn sequences?

We could spend all day tracing our finger around the de Bruijn graph and write down de Bruijn sequences. However, as k gets large the de Bruijn graph becomes very messy and hard to use.

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Donald Knuth's Art of Computer Programming Volume 4 [3] has an entire section on algorithms for finding de Bruijn sequences. If you are interested this is good place to get information.

Our Special de Bruijn Sequence

Recall that the de Bruijn sequence for our trick is

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Question

Once we know the first card how do we determine the next four?

$a + c$ to the end

If the current window is *abcde*, where a, b, c, d, and e are all binary digits in the sequence then the next sequence is $bcde(a + c)$.

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Note: This method of working through the sequence does not include 00000, which is the 8 of Clubs.

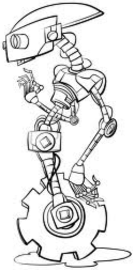
Outline

de Bruijn Sequences, not just for magic anymore

Some uses of de Bruijn sequences:

- Robotics
- Data Encryption
- DNA fragment assembly

Where Am I?



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de Bruijn Arrays

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- If so, is there a nice way to produce them?
- How many are there?
- Are there always de Bruijn Arrays that allow for easy determination of location?

de Bruijn Arrays

- Do de Bruijn Arrays always exist? In special cases yes, but in general we don't know.
- If so, is there a nice way to produce them? Not sure.
- How many are there? Not sure.
- Are there always de Bruijn Arrays that allow for easy determination of location? Not sure.

Digital Pens and Paper

Livescribe [2] Dot Paper that uses the Anoto Pattern

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- If one thinks of a dot as being a 1 and a blank space as being a 0. This is a sort of de Bruijn array of dots.

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- There are other such "digital papers" out there.

Outline

The Mathematician as an Explorer

yamátárájabhánasalagám [6]

The Mathematician as an Explorer

yamátárájabhánasalagám [6]

0111010001

Thank You

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