Avoidance of Partitions of a Three Element Set

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Introduction

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History

- Noncrossing Partitions of Kreweras (1972)
- Klazar (1996, 2000) first to give a definition
- Sagan (2006)
- Klazar and Marcus (2006)

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Definitions

Definition

A partition π of a set S, written $\pi \vdash S$, is $B_1, B_2, \ldots B_k \subseteq S$, called blocks, so that $\biguplus_{i=1}^{k} B_k = S$.

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$$\pi = B_1/B_2/\ldots/B_k$$

Canonical Ordering: $\min B_1 < \min B_2 < \cdots < \min B_k$

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$\Pi_n := \{\pi \vdash [n]\}$ and $[n] := \{1, 2, \dots, n\}$

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Example

$137/26/45 \vdash [7]$

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A subpartition of a partition σ*, is a partition* σ' *so that each block of* σ' *is contained in a different block of* σ*.*

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Definition

A *subpartition* *of a partition* σ*, is a partition* σ' *so that each block of* σ' *is contained in a different block of* σ*.*

Example

$137/26/45 \vdash [7]$
$26/3$ *is a subpartition,*

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Definitions

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Definition

A subpartition of a partition σ*, is a partition* σ' *so that each block of* σ' *is contained in a different block of* σ*.*

Example

$137/26/45 \vdash [7]$

$26/3$ *is a subpartition, but 13/2/6 is not.*

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Definitions

Definition

Let $S \subseteq \mathbb{Z}$ and $\#S = n$ then the standardization map is the unique order preserving bijection $St_S : S \to [n]$.

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Definitions

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Let $S \subseteq \mathbb{Z}$ and $\#S = n$ then the standardization map is the unique order preserving bijection $St_S : S \to [n]$.

Example

$S = \{2, 3, 5, 8, 9\}$ *then*

$$\begin{array}{rcl} St : S & \to & [5] \\ 2 & \mapsto & 1 \\ 3 & \mapsto & 2 \\ & etc & \end{array}$$

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$$\begin{array}{rcl} St : S & \to & [5] \\ 2 & \mapsto & 1 \\ 3 & \mapsto & 2 \\ & etc & \end{array}$$

St acts elmentwise.
38925

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Example

$S = \{2, 3, 5, 8, 9\}$ *then*

$$\begin{aligned} St : S &\to [5] \\ St : 3 &\mapsto 1 \\ St : 7 &\mapsto 2 \\ &etc \end{aligned}$$

St acts elmentwise.
$St(38925) = 24513$

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Definitions

Definition

Let $p = p_1 p_2 \ldots p_k \in S_k$ and $q = q_1 q_2 \ldots q_n \in S_n$. Then q contains the pattern p if there exists $q' = q_{i_1} q_{i_2} \ldots q_{i_k}$ with $St(q') = p$. Otherwise, q avoids p.

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Example

$\sigma = 137/26/45$

Pattern: $\pi = 13/2$

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Example

$\sigma = 137/26/45$
Pattern: $\pi = 13/2$
Copies: $\pi' = 17/4$ *or* $26/3$

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Definitions

Definition

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Definition

Let $\pi \in \Pi_k$ *and* $\sigma \in \Pi_n$. *Then* σ *contains the pattern* π *if there is a subpartition* σ' *of* σ *with* $St(\sigma') = \pi$. *Otherwise,* σ *avoids* π.

Example

$\sigma = 137/26/45$
Pattern: $\pi = 13/2$
Copies: $\pi' = 17/4$ *or* $26/3$
σ *avoids 1/234.*

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Enumeration

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Definition

For a subset $R \subseteq S_k$ and a subset $S \subseteq \Pi_k$ let

$$S_n(R) = \{q \in S_n : q \text{ avoids every } p \in R\},$$

$$\Pi_n(S) = \{\sigma \in \Pi_n : \ \sigma \text{ avoids every } \pi \in S\}.$$

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Question

For any $R \subseteq \Pi_3$ what is $\#\Pi_n(R)$?

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Question

For any $R \subseteq \Pi_3$ what is $\#\Pi_n(R)$?

Theorem (Sagan, 2006)

$$\#\Pi_n(13/2) = 2^{n-1}$$

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Definition

$\sigma \in \Pi_n$ is *layered* if $\sigma = 12\ldots i/i+1\ldots j/\ldots/k+1\ldots n$.

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Example

123/45/6/78

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Definition

$\sigma \in \Pi_n$ *is* *layered* *if* $\sigma = 12\ldots i/i+1\ldots j/\ldots/k+1\ldots n$.

Example

123/45/6/78

Definition

A *composition* *of an integer* n *is an ordered family* $n_1, n_2, \ldots, n_k$ *such that* $n_1 + n_2 + \cdots + n_k = n$. *The* n_i *are called* *parts*.

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Theorem (Sagan, 2006)

$$\#\Pi_n(13/2) = 2^{n-1},$$
$$\Pi_n(13/2) = \{\sigma \in \Pi_n : \sigma \textit{ is layered}\}.$$

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Theorem (Sagan, 2006)

$$\#\Pi_n(13/2) = 2^{n-1},$$

$$\Pi_n(13/2) = \{\sigma \in \Pi_n : \sigma \textit{ is layered}\}.$$

Proof: Let $\sigma = B_1/B_2/\ldots/B_k \in \Pi_n(13/2)$, and let $m = \max B_1$. By the canonical ordering $1 = \min B_1$. If $1 < a < m$ and $a \notin B_1$ then $1m/a$ is a copy of $13/2$. Thus, $B_1 = \{1, 2, \ldots, m\}$. Repeat this process for the remaining blocks to see that σ must be layered.

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The number of compositions of n is known to be 2^{n-1}. There is an easy bijection between layered partitions and compositions of n, given by
$B_1/B_2/\ldots/B_k \mapsto n_1 + n_2 + \cdots + n_k$ with $\#B_i = n_i$. $\diamond$

Definition

We say that $\sigma = B_1/B_2/\ldots/B_k$ *is a* *matching* *if* $\#B_i \leq 2$ *for all* i*.*

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Definition

We say that $\sigma = B_1/B_2/\dots/B_k$ *is a* *matching* *if* $\#B_i \leq 2$ *for all* i.

Theorem (G., 2006)

$$\begin{aligned}
\#\Pi_n(13/2, 123) &= F_n, \\
\Pi_n(13/2, 123) &= \{\sigma = B_1/B_2/\dots/B_k \in \Pi_n : \\
&\qquad \sigma \textit{ is a layered matching}\}.
\end{aligned}$$

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Theorem (G., 2006)

$$\begin{aligned} \#\Pi_n(13/2, 123) &= F_n, \\ \Pi_n(13/2, 123) &= \{\sigma = B_1/B_2/\dots/B_k \in \Pi_n : \\ &\qquad \sigma \textit{ is a layered matching}\}. \end{aligned}$$

Proof: These are in bijection with compositions of n with part sizes at most 2, which are known to be counted by the Fibonacci numbers. $\diamond$

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Generalized Partition Patterns

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History and Notation

Babson and Steingrímsson (2000).

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History and Notation

Babson and Steingrímsson (2000).

Always write $\sigma = B_1/B_2/\ldots/B_k$ so that

$$\min B_1 < \min B_2 < \cdots < \min B_k.$$

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Always write $\sigma = B_1/B_2/\ldots/B_k$ so that

$$\min B_1 < \min B_2 < \cdots < \min B_k.$$

Consider the pattern $\pi = 13/2$.

Two ways to generalize this:

- Adjacent Blocks: Pattern $13|2$
- Adjacent Elements: Pattern $\overset{\frown}{13}/2$

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Example

Consider the partition $\sigma = 137/26/45$.

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Example

Consider the partition $\sigma = 137/26/45$.

Then $17/2$ *is a copy of* $13|2$, *but* $17/4$ *is not,*

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Example

Consider the partition $\sigma = 137/26/45$.

Then $17/2$ *is a copy of* $13|2$, *but* $17/4$ *is not,*

and $13/2$ *is a copy of* $\overset{\frown}{13}/2$, *but* $17/2$ *is not.*

Example

Consider the partition $\sigma = 137/26/45$.

Then $\color{blue}{17/2}$ *is a copy of* $13|2$, *but* $\color{red}{17/4}$ *is not,*

and $\color{blue}{13/2}$ *is a copy of* $\overset{\frown}{13}/2$, *but* $\color{red}{17/2}$ *is not.*

Also, $\color{blue}{26/3}$ *is a copy of* $13|2$.

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Lemma

The following are true for generalized partitions:

$$
\begin{array}{rcccccc}
\Pi_n(1/2/3) & = & \Pi_n(1|2/3) & = & \Pi_n(1/2|3) & = & \Pi_n(1|2|3), \\
\Pi_n(1/23) & = & \Pi_n(1|23) & = & \Pi_n(1/\overset{\frown}{23}) & = & \Pi_n(1|\overset{\frown}{23}), \\
\Pi_n(13/2) & = & \Pi_n(\overset{\frown}{13}/2) & = & \Pi_n(13|2) & = & \Pi_n(\overset{\frown}{13}|2), \\
\Pi_n(123) & = & \Pi_n(\overset{\frown}{12}3) & = & \Pi_n(1\overset{\frown}{23}) & = & \Pi_n(\overset{\frown}{1\,2}\overset{\frown}{\,3}), \\
\Pi_n(12/3) & = & \Pi_n(\overset{\frown}{12}/3), & & & & \\
\Pi_n(12|3) & = & \Pi_n(\overset{\frown}{12}|3). & & & &
\end{array}
$$

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Proposition

Let $S \subseteq \Pi_3 - \{12/3\}$ *then* $\Pi_n(S \cup \{12|3\}) = \Pi_n(S \cup \{12/3\})$ *unless* $S = \emptyset$ *or* $\{123\}$.

Proposition

Let $S \subseteq \Pi_3 - \{12/3\}$ *then* $\Pi_n(S \cup \{12|3\}) = \Pi_n(S \cup \{12/3\})$ *unless* $S = \emptyset$ *or* $\{123\}$.

Lemma

We have that $\sigma \in \Pi_n(12|3)$ *if and only if whenever a block* B_t *of* σ *satisfies* $\#B_t \geq 2$, *then*

$$\#B_{t-1} = 1 \text{ and } \#B_{t+1} = 1.$$

Furthermore, if $B_{t+1} = \{a\}$ *then* $a < b$ *for every* $b \in B_t - \{\min B_t\}$.

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Definition

Let $a_n = \#\Pi_n(12|3)$ *and* $f(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$ *be the corresponding exponential generating function.*

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Definition

Let $a_n = \#\Pi_n(12|3)$ *and* $f(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$ *be the corresponding exponential generating function.*

Proposition

We have that

$$a_n = a_{n-1} + 1 + \sum_{k=1}^{n-2} \binom{n-2}{k} a_{n-k-2}$$

with the initial conditions $a_0 = 1$ *and* $a_1 = 1$*, and* $f(x)$ *satisfies the differential equation*

$$y'' = y' + y(e^x - 1) + e^x.$$

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The substitution $y = ue^{x/2}$ simplifies the equation to

$$u'' = u(e^x - \frac{3}{4}) + e^{x/2}.$$

Using Maple we obtain the solution

$$u = C_1 \cdot I_{\sqrt{-3}}(2e^{x/2}) + C_2 \cdot K_{\sqrt{-3}}(e^{x/2}) +$$

$$2I_{\sqrt{-3}}(2e^{x/2}) \int K_{\sqrt{-3}}(e^{x/2})dx -$$

$$2K_{\sqrt{-3}}(e^{x/2}) \int I_{\sqrt{-3}}(2e^{x/2})e^{x/2}dx,$$

where $I_n(z)$ and $K_n(z)$ are the modified Bessel functions of the first and second kind respectively.

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Set Partition Statistics

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History and Definitions

- Carlitz (1933, 1948) and Gould (1961) introduced two different q-Stirling numbers of the second kind.
- Milne (1978) determined an inversion statistic whose distribution is given by these q-Stirling numbers of the second kind.
- Sagan (1991) shows that a maj statistic for set partitions also has this distribution.
- Wachs and White (1991) studied four statistics, which are variants of the inversion statistic of Milne and also have the q-Stirling numbers of the second kind distribution.

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History and Definitions

- Carlitz (1933, 1948) and Gould (1961) introduced two different q-Stirling numbers of the second kind.
- Milne (1978) determined an inversion statistic whose distribution is given by these q-Stirling numbers of the second kind.
- Sagan (1991) shows that a maj statistic for set partitions also has this distribution.
- Wachs and White (1991) studied four statistics, which are variants of the inversion statistic of Milne and also have the q-Stirling numbers of the second kind distribution.

Question

Can these statistics be written in terms of generalized patterns?

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Definition

Let $\sigma = B_1/B_2/\ldots/B_k$, *and* $b \in B_j$ *then* (b, B_i) *is*

- *an* *inversion* *if* $j < i$ *and* $b > \min B_i$,
- *a* *descent* *if* $j = i - 1$ *and* $b > \min B_i$. *Let* d_i *be the number of descents of* σ *of the form* (b, B_i).

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Example

If $\sigma = 137/26/45$ *then* $(7, \{4, 5\})$ *is an inversion since* $7 > 4$*, and* $(3, \{2, 6\})$ *is a descent, since 3 appears in the block immediately before* $\{2, 6\}$*.*

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Definition

The inversion statistic, $inv(\sigma)$*, is the number of inversions of* σ *and the major index,*
$maj(\sigma) = \sum id_i = d_1 + 2d_2 + \cdots + kd_k$*.*

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We will treat a generalized pattern ρ as a function $\rho : \bigcup_{n\geq 1} \Pi_n \to \mathbb{Z}$ where $\rho(\sigma)$ is the number of copies of ρ in σ.

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Theorem (G. 2006)

For any $\sigma \in \Pi_n$

$$\begin{aligned} inv(\sigma) &= (\overset{\frown}{13}/\overset{\frown}{2})(\sigma) \\ maj(\sigma) &= (\overset{\frown}{13}|\overset{\frown}{2}+\overset{\frown}{1}/\overset{\frown}{24}|\overset{\frown}{3}). \end{aligned}$$

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Question

How are some of these statistics distributed on the sets $\Pi_n(13/2)$ *and* $\Pi_n(13/2, 123)$*?*

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Observation

Only the statistics ls and rb of Wachs and White give nontrivial distributions over $\Pi_n(13/2)$ and $\Pi_n(13/2, 123)$.

Question

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Definition

Let the q-analogue $F_n(q)$ of the Fibonacci numbers be given by the recursion

$$F_n(q) = q^{n-1}F_{n-1}(q) + q^{n-2}F_{n-2}(q)$$

with initial conditions $F_0(q) = 1$ and $F_1(q) = 1$.

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Theorem (G.-Sagan)

$$\sum_{\pi \in \Pi_n(13/2)} q^{ls(\pi)} = \sum_{\pi \in \Pi_n(13/2)} q^{rb(\pi)} = \prod_{k=1}^{n-1} (1 + q^k),$$

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$$\sum_{\pi\in\Pi_n(13/2,123)} q^{ls(\pi)} = \sum_{\pi\in\Pi_n(13/2,123)} q^{rb(\pi)} = F_n(q).$$

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Related to q-Fibonacci numbers of Carlitz (1974,1975) and Cigler (2003,2004).

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Related to q-Fibonacci numbers of Carlitz (1974,1975) and Cigler (2003,2004).

Well known identity for the Fibonacci numbers:

$$F_n = \sum_{2k \leq n} \binom{n-k}{k}.$$

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Related to q-Fibonacci numbers of Carlitz (1974,1975) and Cigler (2003,2004).

Well known identity for the Fibonacci numbers:

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Theorem (Carlitz, G.-Sagan)

$$F_n(q) = \sum_{2k \leq n} q^{\binom{n}{2}+k^2-nk} \begin{bmatrix} n-k \\ k \end{bmatrix}_q .$$

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Future Work

- Avoidance of both patterns from Π_3 and from Π_4.
- Avoidance of Infinite families of patterns (Sagan, 2006).
- Questions of Containment and Packing analogous to work initiated by (Price, 1997) for permutations.
- Is there a way to classify statistics giving rise to the q-Stirling numbers of the second kind as Babson and Steingrímsson did for Mahonian statistics?
- What sort of distributions arrise from studying the aforementioned statistics on other restricted sets of partitions? (Currently being studied by G.)
- The definition of containment of a set partition produces a nice partial ordering. What kind of structure does this poset have? (Currently being investigated by G.)

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Thank You!