Pattern Avoidance in Ordered Set Partitions

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Outline

Set Partitions

Definition

Let $[n] = \{1, 2, \ldots, n\}$ *a* *partition,* π*, of* $[n]$, $\pi \vdash [n]$ *is a family of sets* $B_1, B_2, \ldots, B_k$*, called blocks, such that* $B_1 \uplus B_2 \uplus \cdots \uplus B_k = [n]$*. We write*

$$\pi = B_1/B_2/\ldots/B_k,$$

where $\min B_1 < \min B_2 < \cdots < \min B_k$.

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Example

$\pi = 13/247/56 \vdash [7]$.

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Let $\mathcal{OP}_{n,k}$ *be the set of ordered partitions of* $[n]$ *with* k *blocks, and* $\mathcal{OP}_{n,[b_1,b_2,\ldots,b_k]}$ *be the set of ordered partitions of* $[n]$ *with* k *blocks and* $|B_i| = b_i$.

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Notice that the set of permutations of $[n]$, $\mathcal{S}_n$, is in bijection with $\mathcal{OP}_{n,n} = \mathcal{OP}_{n,[1,1,\ldots,1]}$.

What we came to discuss

Definition

An ordered partition $\sigma = B_1/B_2/\ldots/B_k \in \mathcal{OP}_{n,k}$ *is said to contain a copy of a permutation* $p = p_1 p_2 \cdots p_m$ *if there is a sequence of elements* $a_{i_1} a_{i_2} \cdots a_{i_m}$ *with* $a_{i_j} \in B_{i_j}$ *for* $1 \le j \le m$ *with* $i_1 < i_2 < \cdots < i_m$ *such that* $a_{i_1} a_{i_2} \ldots a_{i_m}$ *is order isomorphic to* p.

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For example let $\pi = 56/247/13$*.* 523 *forms a copy of* 312*, and* 573 *forms a copy of* 231*.*

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Example

For example let $\pi = 56/247/13$. 523 *forms a copy of* 312, *and* 573 *forms a copy of* 231. *However,* π *avoids the permutations* 123 *and* 132.

Outline

$\mathcal{OP}_{n,k}(p)$

Definition

Let $\mathcal{OP}_{n,k}(p)$ *be the set of partitions in* $\mathcal{OP}_{n,k}$ *avoiding the permutation* p*, and* $op_{n,k}(p) = |\mathcal{OP}_{n,k}(p)|$.

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Theorem

For $n \geq 1$,

$$op_{n,k}(12) = \binom{n-1}{k-1}.$$

Avoiding a Pattern of Length 3

Theorem

For $n \geq 1$ and $p \in \mathcal{S}_3$,

$$op_{n,3}(p) = \left(\frac{n^2}{8} + \frac{3n}{8} - 2\right) 2^n + 3$$

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Conjecture

For $n \geq 1$, n even, and $p \in \mathcal{S}_3$, we have that

$$op_{n,[2,2,\ldots,2]} \sim \sqrt{12}^n.$$

Partitions with $n-1$ blocks

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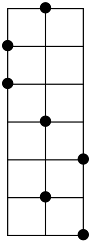
Theorem

$$\begin{aligned}
\widehat{op}_{n,n-1} &= \sum_{k=0}^{n} C_k + \sum_{a=2}^{n-1}\sum_{k=0}^{n-a-2} C_k \sum_{i=1}^{a-1} r(a-1,i)\binom{i+n-k-a-1}{i} \\
&+ \sum_{a=2}^{n-1} C_{n-a-1}C_{a-1} + \sum_{a=2}^{n-1}\sum_{k=n-a}^{n-2} C_{n-a-1}C_{k-n+a+1}C_{n-k-2} \\
&+ 2\sum_{j=1}^{n} C_{j-1}\widehat{op}_{n,n-j}.
\end{aligned}$$

Outline

Graphs of Ordered Partitions

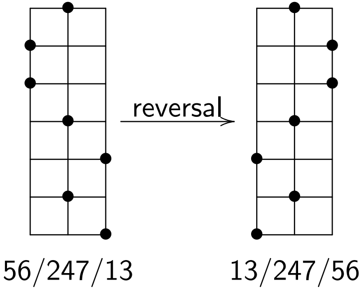
A graph of the partition 56/247/13.



56/247/13

The graph of an ordered partition is like that of a permutation, we simply allow more than one entry in each column.

Graphs of Ordered Partitions and Symmetries



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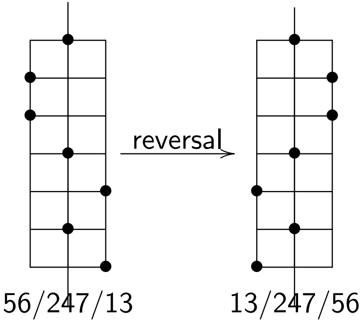
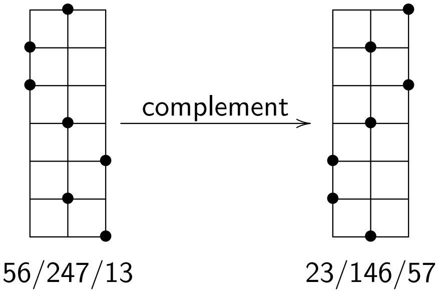


Of course the reversal map is just a reflection over the vertical axis that divides the partition graph in half.

Graphs of Ordered Partitions and Symmetries



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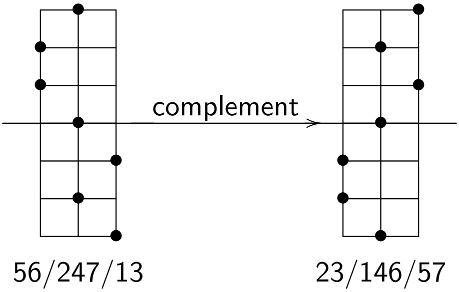

Of course the complement map is just a reflection over the horizontal axis that divides the partition graph in half.

Graphs of Ordered Partitions and Symmetries

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The ordered partition $\pi \in \mathcal{OP}_{n,k}$ avoids permutation $p \in \mathcal{S}_m$ iff π^r avoids p^r and π avoids p iff π^c avoids p^c. $\square$

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This gives us that $\mathrm{op}_{n,k}(123) = \mathrm{op}_{n,k}(321)$ and $\mathrm{op}_{n,k}(132) = \mathrm{op}_{n,k}(231) = \mathrm{op}_{n,k}(213) = \mathrm{op}_{n,k}(312)$, as with permutations.

$\text{op}_{n,[b_1,b_2,\ldots,b_k]}(123) = \text{op}_{n,[b_1,b_2,\ldots,b_k]}(132)$

Theorem

For $n \geq 1$ and any $b_1, b_2, \ldots, b_k$ with $\sum b_i = n$ we have,

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The proof is an adaptation of the bijective proof of this very same fact for permutations given by Simion and Schmidt. To do this we need to define what a left-to-right minimum is in an ordered partition.

Left-To-Right Minima

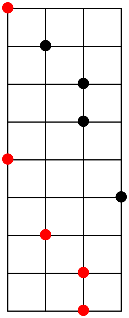
Definition

Let $\pi = B_1/B_2/\ldots/B_k \in \mathcal{OP}_{n,k}$. The element $a \in B_i$ is a left-to-right minimum of π if $a \leq b$ for any $b \in B_j$ with $j < i$.

$\text{op}_{n,[b_1,b_2,\ldots,b_k]}(123) = \text{op}_{n,[b_1,b_2,\ldots,b_k]}(132)$

Example

Consider the ordered partition $59/38/1267/4$. *The left-to-right minima are* $1, 2, 3, 5, 9$.



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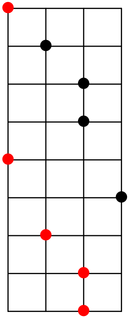

We observe that
- Every element in the first block is a left to right minimum.

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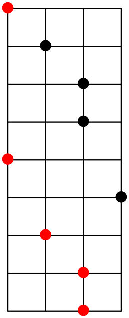

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- The sequence of left to right minima is "decreasing".

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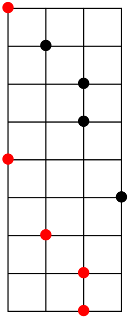

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- Every element in the first block is a left to right minimum.

- The sequence of left to right minima is "decreasing".

- If $\pi \in \mathcal{OP}_{n,[b_1,b_2,\ldots,b_k]}(123)$ then the remaining elements form a decreasing sequence.

The Bijection (a la Simion and Schmidt)

Let $\phi : \mathcal{OP}_{n,[b_1,b_2,\ldots,b_k]}(123) \to \mathcal{OP}_{n,[b_1,b_2,\ldots,b_k]}(132)$ by

- Fix all left-to-right minima in their blocks, and remove the remaining entries from the partition.

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- Replace the elements in the blocks from left to right by placing the smallest remaining element that is larger than the preceeding left-to-right minimum.

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- Replace the elements in the blocks from left to right by placing the smallest remaining element that is larger than the preceeding left-to-right minimum.
- The new partition will be 132 avoiding.

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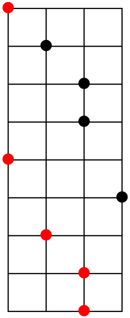

59/38/1267/4

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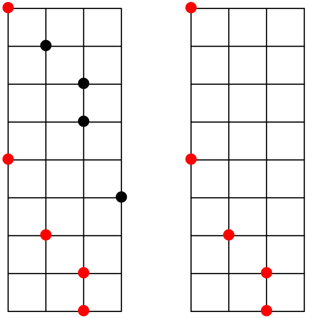

59/38/1267/4   59/3/12/

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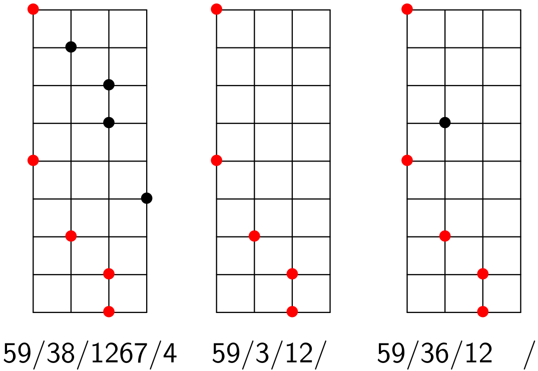

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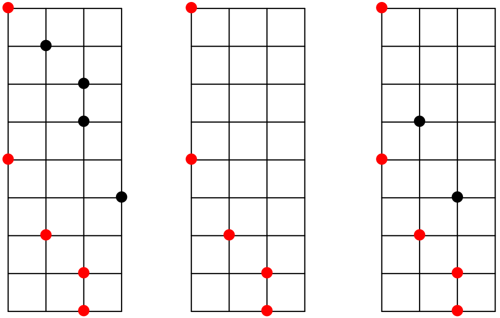

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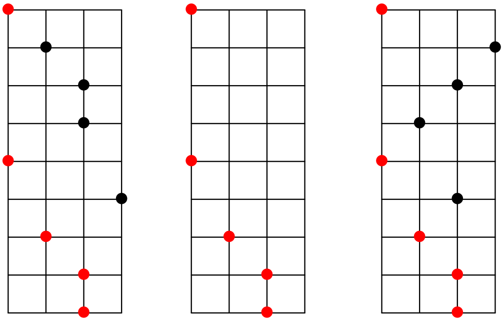

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The inverse of this operation is to fix the left-to-right minima and replace the missing elements in decreasing order.

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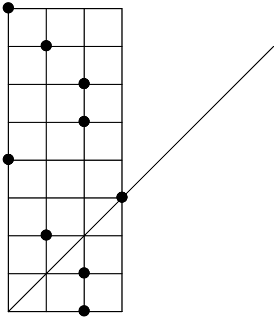
Theorem

$\phi : \mathcal{OP}_{n,[b_1,b_2,\ldots,b_k]}(123) \to \mathcal{OP}_{n,[b_1,b_2,\ldots,b_k]}(132)$ *is a bijection.*

Outline

The Inverse Symmetry

Consider the Graph of the ordered partition 59/38/1267/4.



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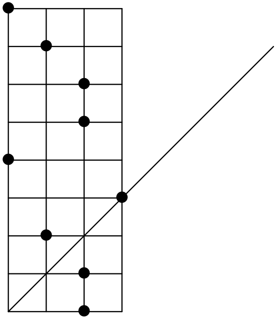

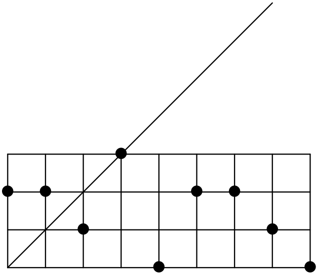

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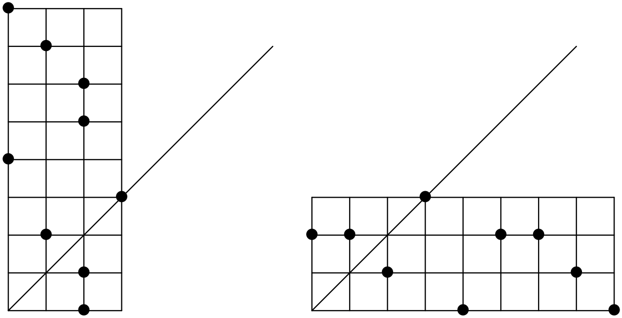

Not an Ordered Partition!

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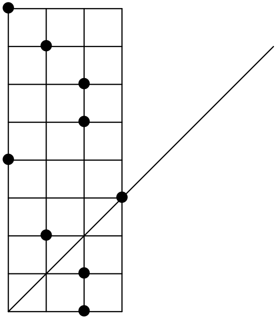

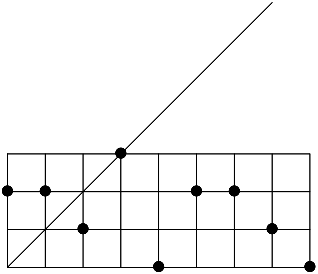

Not an Ordered Partition!
But it is a Word!

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Let p *be a permutation and* $[k]^n(p)$ *be the set of words avoiding* p*.*

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Let p be a permutation and $[k]^n(p)$ be the set of words avoiding p.

Example

The word $443624432 \in [6]^9$ *avoids* 123.

A Familiar Theorem

Theorem (Burstein, Bränden-Mansour, Jelínek-Mansour)

$|[k]^n(123)| = |[k]^n(132)|$

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Proof:

- Notice that $123^i = 123$ and $132^i = 132$, so if π avoids 123 then so does π^i.

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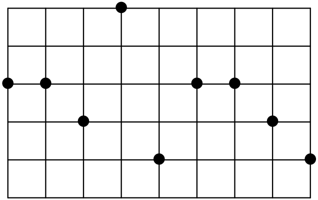
- Notice that $123^i = 123$ and $132^i = 132$, so if π avoids 123 then so does π^i.
- The map $i \circ \phi \circ i : [k]^n(123) \to [k]^n(132)$ is a bijection.

An Example

Consider the word 443624432, with graph



443624432

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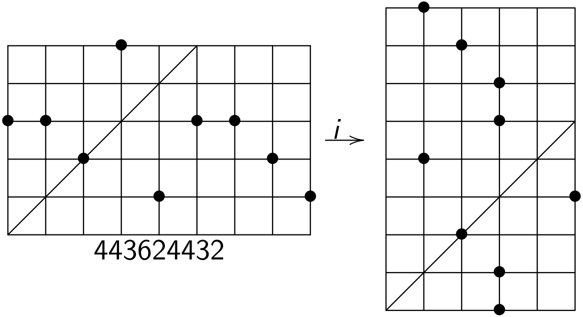

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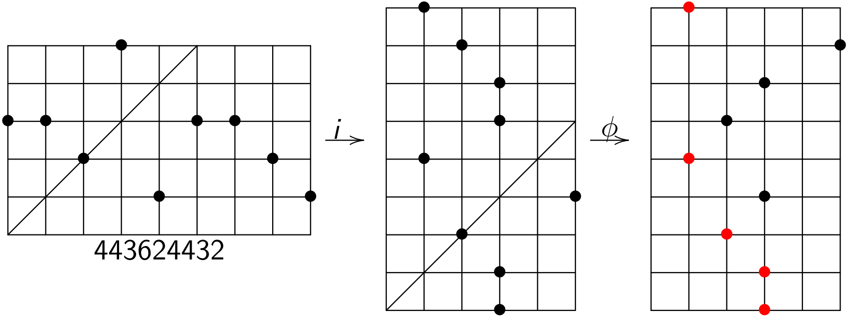

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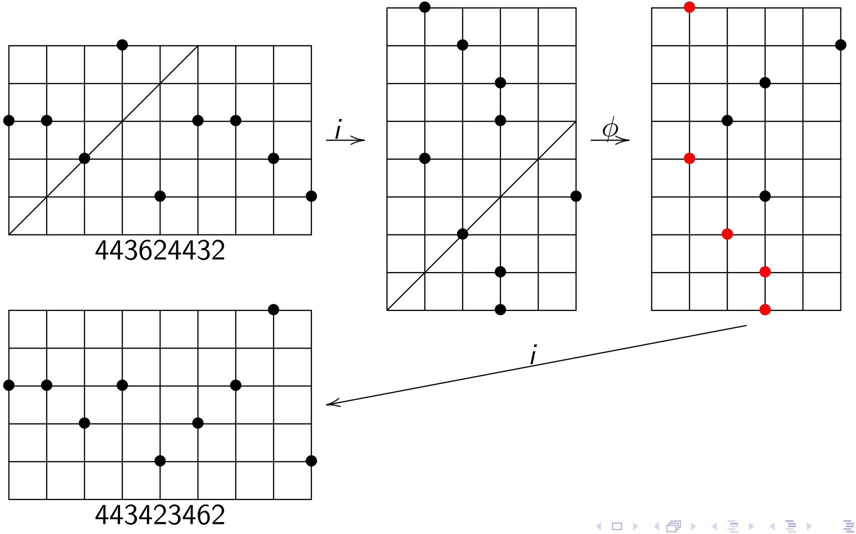

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- Packing?
- So many problems to solve, it's like Christmas!

Thank You

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