Packing Set Partitions

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Outline

History

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- Burstein, Hästö, and Mansour [2] determine packing densities for words.
- Three years ago I am introduced to packing and am encouraged to consider packing for set partitions.

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Example

$137/25/46 \vdash [7]$

Canonical Words

To each set partition is associated a canonical word $a_1 a_2 \ldots a_n$ where $a_i = j$ if $i \in B_j$.

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137/25/46 *corresponds to* 1213231.

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Given any word $w \in [k]^n$ we may canonize w by replacing all occurrences of the first letter of w by 1, all occurrences of the next occurring letter by 2, etc.

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From now on we will refer to these canonical words as partitions.

Order Isomorphism

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We say that two words $\alpha = \alpha_1\alpha_2\ldots\alpha_n$ *and* $\beta = \beta_1\beta_2\ldots\beta_n$ *are* ***order isomorphic*** *if for* $i < j$

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Example

The words 12332 *and* 51771 *are not order isomorphic.*

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1. *Non-Restricted:*
   there is a subsequence of σ of length k whose canonization is π.
2. *Restricted:*
   there is a subsequence of σ of length k that is order isomorphic to π.

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And 131*, as in* 1213231*, is a copy of 121 in both the restricted and non-restricted sense.*

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Example

Let $\sigma = 1213231$ *and* $\pi = 121$*.*

Then 212*, as in* 1213231 *is a copy of 121 in the non-restricted sense, but not in the restricted sense.*

And 131*, as in* 1213231*, is a copy of 121 in both the restricted and non-restricted sense.*

The non-restricted pattern containment definition allows for the blocks of the copy to be in a different order than the blocks in the pattern. The restricted definition does not.

Packing Definitions

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Density

Theorem

For $n \geq m$ we have that $d(E, k, n-1) \geq d(E, k, n)$ and $d(E, k, n) \geq d(E, k-1, n)$. $\square$

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Definition

Let the packing density of the set $E \subseteq \Pi_m$ be

$$\delta(E) = \lim_{n\to\infty} d(E, n, n) = \lim_{n\to\infty} \lim_{k\to\infty} d(E, k, n).$$

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Restricted Results

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- If E consists entirely of layered partitions then there exists a layered σ so that $\nu(E,\sigma) = \mu(E,n,n)$.
- All of the results of Price [3] and Albert, Atkinson, Handley, Holton, and Stromquist [1] about layered permutations apply.
- $\delta(121) = \frac{2\sqrt{3}-3}{2} \approx 0.2321$

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- For example $\pi = 112223334444 4$ is monotone layered, but $\pi = 112333$ is not monotone layered.
- If E consists entirely of monotone layered partitions then there exists some σ, which is also monotone layered, such that $\nu(E, \sigma) = \mu(E, n, n)$.

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- For example $\pi = 1122233344444$ is monotone layered, but $\pi = 112333$ is not monotone layered.
- If E consists entirely of monotone layered partitions then there exists some σ, which is also monotone layered, such that $\nu(E, \sigma) = \mu(E, n, n)$.
- Thus, all results for layered permutations that are monotone apply.

Monotone Layered Theorem

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Theorem

Given a monotone increasing layered pattern $\pi = 1 \ldots 12 \ldots 2 \ldots k \ldots k$ then among all partitions with the same block structure as σ, the layered monotone increasing one is a maximizer.

Proof

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The last letter of σ was the last occurrence of some j in the partition.
Let $\hat{j}$ be the letter corresponding to j in $\hat{\sigma}$.
By induction on the length of σ the number of copies of π in σ that do not involve this j is no more than the number of copies in $\hat{\sigma}$ that do not involve $\hat{j}$.

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Let $\hat{j}$ be the letter corresponding to *j* in $\hat{\sigma}$.
By induction on the length of σ the number of copies of π in σ that do not involve this *j* is no more than the number of copies in $\hat{\sigma}$ that do not involve $\hat{j}$.
Now consider the number of copies of π in σ that did involve this *j*. In any such copy the *j*s must correspond to the *k*s of π. By inducting on *k* we can say that number of copies has not decreased in this case as well. □

121

Conjecture

The partition of $[n]$ *that maximizes the number of copies of* 121 *is* $\underbrace{121212\ldots}_{n}$

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Example

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Theorem

$\delta(121) \leq \frac{1}{2}$. $\Box$

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- Can it be shown that 12121 . . . is the best possible for packing 121?
- Can we at least get a better upper bound for 121?
- Anything that might translate from the permutation case to the set partition case.

THANK YOU

- M. H. Albert, M. D. Atkinson, C. C. Handley, D. A. Holton, W. Stromquist, On packing densities of permutations, Electron. J. Combin. 9 (1) (2002) Research Paper 5, 20 pp. (electronic).
- A. Burstein, P. Hästö, T. Mansour, Packing patterns into words, Electron. J. Combin. 9 (2) (2002/03) Research paper 20, 13 pp. (electronic), permutation patterns (Otago, 2003).
- A. Price, Packing densities of layered patterns, Ph.D. thesis, University of Pennsylvania, Philadelphia, PA, 1997.