### **Packing Set Partitions**

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Results for the Non-Restricted Definition



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### Outline



Results for the Restricted Definition

- 3 Results for the Non-Restricted Definition
- Open Problems

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- Three years ago I am introduced to packing and am encouraged to consider packing for set partitions.

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#### Example

137/25/46 ⊢ [7]

### **Canonical Words**

# To each set partition is associated a canonical word $a_1 a_2 \dots a_n$ where $a_i = j$ if $i \in B_j$ .

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# Example 137/25/46 *corresponds to* 1213231.

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Given any word  $w \in [k]^n$  we may canonize w by replacing all occurrences of the first letter of w by 1, all occurrences of the next occurring letter by 2, etc.

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From now on we will refer to these canonical words as partitions.

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#### Example

The words 12332 and 51771 are not order isomorphic.

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### Pattern Containment

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Let  $\sigma$  be a partition with  $\ell(\sigma) = n$  and  $\pi$  be a partition with  $\ell(\pi) = k$ . We say that  $\sigma$  contains  $\pi$  if

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#### 2 Restricted:

there is a subsequence of  $\sigma$  of length k that is order isomorphic to

 $\pi$ .

#### Example

*Let*  $\sigma$  = 1213231 *and*  $\pi$  = 121.

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Then 212, as in 1213231 is a copy of 121 in the non-restricted sense, but not in the restricted sense.

And 131, as in 1213231, is a copy of 121 in both the restricted and non-restricted sense.

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#### Example

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Then 212, as in 1213231 is a copy of 121 in the non-restricted sense, but not in the restricted sense.

And 131, as in 1213231, is a copy of 121 in both the restricted and non-restricted sense.

The non-restricted pattern containment definition allows for the blocks of the copy to be in a different order than the blocks in the pattern. The restricted definition does not.

#### Definition

Let  $E \subseteq \prod_m$  and  $\sigma \in \prod$  and  $\nu(E, \sigma)$  be the number of copies of partitions from *E* in  $\sigma$  then define:

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 $\mu(E, k, n) = \max\{\nu(E, \sigma) : \sigma \vdash [n] \text{ and } \sigma \text{ has at most } k \text{ blocks}\}$ 

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### Density

#### Theorem

# For $n \ge m$ we have that $d(E, k, n-1) \ge d(E, k, n)$ and $d(E, k, n) \ge d(E, k-1, n)$ . $\Box$

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# Density

#### Theorem

For  $n \ge m$  we have that  $d(E, k, n-1) \ge d(E, k, n)$  and  $d(E, k, n) \ge d(E, k-1, n)$ .  $\Box$ 

#### Definition

Let the packing density of the set  $E \subseteq \Pi_m$  be

$$\delta(E) = \lim_{n \to \infty} d(E, n, n) = \lim_{n \to \infty} \lim_{k \to \infty} d(E, k, n).$$

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### Outline



### 2 Results for the Restricted Definition

3 Results for the Non-Restricted Definition

### Open Problems

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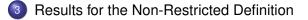
• 
$$\delta(121) = \frac{2\sqrt{3}-3}{2} \approx 0.2321$$

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# Outline







#### 4 Open Problems

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- If *E* consists entirely of monotone layered partitions then there exists some  $\sigma$ , which is also monotone layered, such that  $\nu(E, \sigma) = \mu(E, n, n)$ .
- Thus, all results for layered permutations that are monotone apply.

# Monotone Layered Theorem

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#### Theorem

Given a monotone increasing layered pattern  $\pi = 1 \dots 12 \dots 2 \dots k \dots k$  then among all partitions with the same block structure as  $\sigma$ , the layered monotone increasing one is a maximizer.

**Proof:** Consider  $\hat{\sigma}$ , the monotone increasing layered partition with the same block structure as  $\sigma$ . We must show that  $\nu(\pi, \sigma) \leq \nu(\pi, \hat{\sigma})$ .

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- **Proof:** Consider  $\hat{\sigma}$ , the monotone increasing layered partition with the same block structure as  $\sigma$ . We must show that  $\nu(\pi, \sigma) \leq \nu(\pi, \hat{\sigma})$ . The last letter of  $\sigma$  was the last occurrence of some *j* in the partition.
- Let  $\hat{j}$  be the letter corresponding to j in  $\hat{\sigma}$ .
- By induction on the length of  $\sigma$  the number of copies of  $\pi$  in  $\sigma$  that do not involve this *j* is no more than the number of copies in  $\hat{\sigma}$  that do not involve  $\hat{j}$ .
- Now consider the number of copies of  $\pi$  in  $\sigma$  that did involve this *j*. In any such copy the *j*s must correspond to the *k*s of  $\pi$ . By inducting on *k* we can say that number of copies has not decreased in this case as well.  $\Box$

#### Conjecture

The partition of [n] that maximizes the number of copies of 121 is  $\underbrace{121212...}_{n}$ 

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#### Example

$$\lim_{n\to\infty} d(121,\underbrace{121212...}_{n}) = \frac{1}{4}.$$

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Image: A matrix and a matrix

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#### Theorem

$$\delta(121) \leq \frac{1}{2}$$
.  $\Box$ 

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Image: A matrix

# Outline



Results for the Restricted Definition





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- Can it be shown that 12121... is the best possible for packing 121?
- Can we at least get a better upper bound for 121?
- Anything that might translate from the permutation case to the set partition case.

# THANK YOU

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