Packing Set Partitions: A New Hope

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Introduction

Previous Results

Death Star ($\delta(121) = \frac{1}{4}$) Destruction
Outline

1. Introduction
2. Previous Results
3. Death Star \((\delta(121) = \frac{1}{4})\) Destruction
History

- Wilf suggests people consider packing instead of avoidance for permutations.
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Eight years ago in Iceland I am encouraged to consider packing for set partitions by Walter Stromquist.
Adam Before Electrolysis
Set Partition Definition

Definition

A partition \( \pi \) of a set \( S \), written \( \pi \vdash S \), is a family of disjoint nonempty subsets \( B_i \subset S \), called blocks, such that \( \biguplus B_i = S \).
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A partition $\pi$ of a set $S$, written $\pi \vdash S$, is a family of disjoint nonempty subsets $B_i \subseteq S$, called blocks, such that $\bigcup B_i = S$.

We write

$$\pi = B_1 / B_2 / \ldots / B_k,$$

where

$$\min B_1 < \min B_2 < \cdots < \min B_k,$$
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Example

$137/25/46 \vdash [7]$
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where

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\min B_1 < \min B_2 < \cdots < \min B_k.
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**Example**

137/25/46 \( \vdash [7] \)

Let

\[
\Pi_n = \{ \pi : \pi \vdash [n] = \{1, 2, \ldots, n\}\}, \text{ and } \Pi_{n,k} = \{ \pi \in \Pi_n : \pi \text{ has } k \text{ blocks}\}.
\]
To each set partition is associated a canonical word $a_1 a_2 \ldots a_n$ where $a_i = j$ if $i \in B_j$. 
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**Example**

137/25/46 corresponds to 1213231.
Canonizing Words

Definition

Given any word \( w \in [k]^n \) we may canonize \( w \) by replacing all occurrences of the first letter of \( w \) by 1, all occurrences of the next occurring letter by 2, etc.
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The canonized form of 47411477 is 12133122.
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From now on we will refer to these canonical words as partitions.
Order Isomorphism

Definition

We say that two words $\alpha = \alpha_1 \alpha_2 \ldots \alpha_n$ and $\beta = \beta_1 \beta_2 \ldots \beta_n$ are order isomorphic if for $i < j$

- $\alpha_i < \alpha_j$ if and only if $\beta_i < \beta_j$, and
- $\alpha_i = \alpha_j$ if and only if $\beta_i = \beta_j$. 

Example

The words 12332 and 34774 are order isomorphic.

Example

The words 12332 and 51771 are not order isomorphic.
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Pattern Containment

Definition

Let $\sigma$ be a partition with $\ell(\sigma) = n$ and $\pi$ be a partition with $\ell(\pi) = k$. We say that $\sigma$ contains $\pi$ if
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   there is a subsequence of $\sigma$ of length $k$ whose canonization is $\pi$. 

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Definition

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1. **Non-Restricted:**
   
   there is a subsequence of $\sigma$ of length $k$ whose canonization is $\pi$.

2. **Restricted:**
   
   there is a subsequence of $\sigma$ of length $k$ that is order isomorphic to $\pi$. 

Examples of Containment

Example

Let $\sigma = 1213231$ and $\pi = 121$. 
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Let $\sigma = 1213231$ and $\pi = 121$.

Then 212, as in 1213231 is a copy of 121 in the non-restricted sense, but not in the restricted sense.
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Example

Let $\sigma = 1213231$ and $\pi = 121$.

Then 212, as in 1213231 is a copy of 121 in the non-restricted sense, but not in the restricted sense.

And 131, as in 1213231, is a copy of 121 in both the restricted and non-restricted sense.
Examples of Containment

Example

Let $\sigma = 1213231$ and $\pi = 121$.

Then 212, as in 1213231 is a copy of 121 in the non-restricted sense, but not in the restricted sense.

And 131, as in 1213231, is a copy of 121 in both the restricted and non-restricted sense.

The non-restricted pattern containment definition allows for the blocks of the copy to be in a different order than the blocks in the pattern. The restricted definition does not.
Definition

Let $E \subseteq \Pi_m$ and $\sigma \in \Pi$ and $\nu(E, \sigma)$ be the number of copies of partitions from $E$ in $\sigma$ then define:

$$\mu(E, k, n) = \max \{ \nu(E, \sigma) : \sigma \trianglerighteq [n] \text{ and } \sigma \text{ has at most } k \text{ blocks} \}$$

$$d(E, \sigma) = \nu(E, \sigma)$$

$$d(E, k, n) = \mu(E, k, n)$$
Packing Definitions

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d(E, \sigma) = \frac{\nu(E, \sigma)}{\binom{n}{m}}
\]

\[
d(E, k, n) = \frac{\mu(E, k, n)}{\binom{n}{m}}
\]
Density

**Theorem**

For $n \geq m$ we have that $d(E, k, n - 1) \geq d(E, k, n)$ and $d(E, k, n) \geq d(E, k - 1, n)$. □
Density

Theorem

For $n \geq m$ we have that $d(E, k, n - 1) \geq d(E, k, n)$ and $d(E, k, n) \geq d(E, k - 1, n)$. □

Definition

Let the packing density of the set $E \subseteq \Pi_m$ be

$$
\delta(E) = \lim_{n \to \infty} d(E, n, n) = \lim_{n \to \infty} \lim_{k \to \infty} d(E, k, n).
$$
1. Introduction

2. Previous Results

3. Death Star ($\delta(121) = \frac{1}{4}$) Destruction
These are the results that I presented in Florence 2009.

<table>
<thead>
<tr>
<th>Partition $\pi$</th>
<th>111</th>
<th>112</th>
<th>123</th>
<th>121</th>
</tr>
</thead>
<tbody>
<tr>
<td>Restricted Packing</td>
<td>1</td>
<td>$2\sqrt{3} - 3$</td>
<td>1</td>
<td>$\frac{2\sqrt{3} - 3}{2}$</td>
</tr>
<tr>
<td>Density $\delta_r(\pi)$</td>
<td>1</td>
<td>$2\sqrt{3} - 3$</td>
<td>1</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>Unrestricted Packing</td>
<td>1</td>
<td>$2\sqrt{3} - 3$</td>
<td>1</td>
<td>$\frac{1}{4}$</td>
</tr>
</tbody>
</table>
Outline

1. Introduction
2. Previous Results
3. Death Star ($\delta(121) = \frac{1}{4}$) Destruction
Goal: Show that $121212 \cdots 12$ is a maximizer for $121$. 

1212 \cdots 12$ is a maximizer
Packing 121 into 2 block Partitions

Theorem

The number of copies of 121 in a partition among those with two blocks is maximized by the partition $R2D2 \cdots R2D2$
Packing 121 into 2 block Partitions

Theorem

The number of copies of 121 in a partition among those with two blocks is maximized by the partition $\underbrace{121\cdots121}_n$
Packing 121 into 2 block Partitions

Theorem

The number of copies of 121 in a partition among those with two blocks is maximized by the partition $\underbrace{121}_n \cdots \underbrace{121}_n$

We prove this by proving a series of Lemmas.

Lemma

Let $\pi \in \Pi_{n,2}$ have exactly two blocks. Assume that $\pi$ consists of $i$ 1's and $j$ 2's with $i \geq j$. Then the partition

$$\hat{\pi} = \underbrace{11 \cdots 1}_{\lceil (i-j-1)/2 \rceil} \underbrace{1212 \cdots 121}_{2j+1} \underbrace{11 \cdots 1}_{\lceil (i-j-1)/2 \rceil}$$

satisfies $\nu(121, \hat{\pi}) \geq \nu(121, \pi)$. 
Idea of the Proof

- Suppose that inside of the partition we have a string of length $\ell + 2$ of the form

$$2 1 1 \cdots 1 2$$

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- Suppose that inside of the partition we have a string of length $\ell + 2$ of the form

$$(a \text{ 1's and } b \text{ 2's}) \underbrace{2 \cdots 2}_{\ell} (c \text{ 1's and } d \text{ 2's}).$$

- Suppose that there are $a$ 1’s and $b$ 2’s preceding the initial 2 and $c$ 1’s and $d$ 2’s succeeding the final 2.
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- Suppose that inside of the partition we have a string of length $\ell + 2$ of the form 
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- Suppose that there are $a$ 1's and $b$ 2's preceding the initial 2 and $c$ 1's and $d$ 2's succeeding the final 2.

- Swapping the initial 2 with the first 1 in the run gives us a change of 
  \[ (b + c + \ell - 1) - (a + d + 1) \] copies.
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\[(a \, 1's \text{ and } b \, 2's) \underbrace{2 \, 1 \, 1 \cdots 1 \, 2}_{\ell} \, (c \, 1's \text{ and } d \, 2's).\]

- Suppose that there are $a$ 1's and $b$ 2's preceding the initial 2 and $c$ 1's and $d$ 2's succeeding the final 2.

- Swapping the initial 2 with the first 1 in the run gives us a change of $(b + c + \ell - 1) - (a + d + 1)$ copies.

- Swapping the final 2 with the final 1 in the run gives a change of $(a + d + \ell - 1) - (b + c + 1)$ copies.
Idea of the Proof

- Suppose that inside of the partition we have a string of length $\ell + 2$ of the form
  \[(a\ 1's\ and\ b\ 2's)\ 2\underbrace{11\cdots12}_{\ell} (c\ 1's\ and\ d\ 2's).\]
- Suppose that there are $a$ 1’s and $b$ 2’s preceding the initial 2 and $c$ 1’s and $d$ 2’s succeeding the final 2.
- Swapping the initial 2 with the first 1 in the run gives us a change of $(b + c + \ell - 1) - (a + d + 1)$ copies.
- Swapping the final 2 with the final 1 in the run gives a change of $(a + d + \ell - 1) - (b + c + 1)$ copies.
- So we can perform at least one of these without losing copies of 121.
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- Swapping the initial 2 with the first 1 in the run gives us a change of $(b + c + \ell - 1) - (a + d + 1)$ copies.

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- So we can perform at least one of these without losing copies of 121.

- Repeating this process eventually gives us a middle section of alternating 1's and 2's.
Proof that $12121 \cdots 12$ is maximal in $\Pi_{n,2}$

- Start with a partition $\pi \in \Pi_{n,2}$. 

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Proof that $12121 \cdots 12$ is maximal in $\Pi_{n,2}$

- Start with a partition $\pi \in \Pi_{n,2}$.
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\]

- Changing the last 1 in the run of $a$ 1’s at the beginning and the first 1 in the run of $a$ 1’s at the end, we lose $2ja - j + 2\binom{j+1}{2}$ copies of 121 and gain $2(a - 1)(j + a) + \binom{j+1}{2} + \binom{j+2}{2}$ copies of 121.
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$$\underbrace{11 \cdots 1}_{a} \underbrace{212 \cdots 121}_{2j+1} \underbrace{11 \cdots 1}_{a \text{ or } a-1}$$

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- The case where there are $a - 1$ 1’s at the end gives a net gain of $2(a - 1)^2$ copies of 121.
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- This is a net gain of $a^2 + (a-1)^2$ copies of 121.
- The case where there are $a-1$ 1’s at the end gives a net gain of $2(a-1)^2$ copies of 121.
- Thus the partition $\underbrace{1212 \cdots 121}_{n}$ is a maximizer in $\Pi_{n,2}$
Proof of the case for more than two blocks

Let $\alpha_n = \underbrace{1212 \cdots 12}_n$. 
Proof of the case for more that two blocks

- Let $\alpha_n = \overbrace{1212 \cdots 12}^n$.

- We observe that $\nu(121, \alpha_n) = \frac{1}{24} (n^3 - n)$, and let $g(n) = \frac{1}{24} (n^3 - n)$. 
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- Given any partition $\pi \in \Pi_n$ a copy of 121 in $\pi$ can only use two blocks at a time.

- Assume that $\pi \in \Pi_n$ has 3 blocks. The first has $a$ elements the second $b$ elements and the third $n - a - b$ elements. Then the maximal number of copies in $\pi$ is $g(a + b) + g(n - a) + g(n - b)$. This expression is maximized when $a = b = n/3$. So the number of copies of 121 in $\pi$ is bounded by $3g(2n/3) = \frac{n^3}{27} - \frac{n}{12}$.
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- A similar argument shows that the maximal number of copies in a partition with $k$ blocks is also less that $g(n)$.
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Given any partition \( \pi \in \Pi_n \) a copy of 121 in \( \pi \) can only use two blocks at a time.

Assume that \( \pi \in \Pi_n \) has 3 blocks. The first has \( a \) elements the second \( b \) elements and the third \( n - a - b \) elements. Then the maximal number of copies in \( \pi \) is \( g(a + b) + g(n - a) + g(n - b) \). This expression is maximized when \( a = b = n/3 \). So the number of copies of 121 in \( \pi \) is bounded by \( 3g(2n/3) = \frac{n^3}{27} - \frac{n}{12} \).

A similar argument shows that the maximal number of copies in a partition with \( k \) blocks is also less than \( g(n) \).

Finally, \( \delta(121) = \lim_{n \to \infty} \frac{g(n)}{\binom{n}{3}} = \frac{1}{4} \).
Death Star Explosion

The End of the Death Star
Ur Oh!
Ur Oh!

Thank you in Wookiee.
A special thanks to

- Christopher Lippay
- Julie Lippay, and
- Trina Spaeth
URL http://www.combinatorics.org/Volume_9/Abstracts/v9i1r5.html

URL http://www.combinatorics.org/Volume_9/Abstracts/v9i2r20.html