Packing Set Partitions: A New Hope

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1. Introduction
2. Previous Results
3. Death Star ($\delta(121) = \frac{1}{4}$) Destruction

Outline

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- Layered pattern density is determined for particular layered permutations. Work of Price [3] and Albert, Atkinson, Handley, Holton, and Stromquist [1].
- Burstein, Hästö, and Mansour [2] determine packing densities for words.
- Eight years ago in Iceland I am encouraged to consider packing for set partitions by Walter Stromquist.

Walt Stromquist

Adam Before Electrolysis

Set Partition Definition

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Let

$\Pi_n = \{\pi : \pi \vdash [n] = \{1, 2, \dots, n\}\}$, and $\Pi_{n,k} = \{\pi \in \Pi_n : \pi$ has k blocks.$\}$.

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$137/25/46$ *corresponds to* 1213231.

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Given any word $w \in [k]^n$ we may canonize w by replacing all occurrences of the first letter of w by 1, all occurrences of the next occurring letter by 2, etc.

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From now on we will refer to these canonical words as partitions.

Order Isomorphism

Definition

We say that two words $\alpha = \alpha_1\alpha_2\ldots\alpha_n$ *and* $\beta = \beta_1\beta_2\ldots\beta_n$ *are order isomorphic if for* $i < j$

- $\alpha_i < \alpha_j$ *if and only if* $\beta_i < \beta_j$*, and*
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Example

The words 12332 *and* 51771 *are not order isomorphic.*

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1. *Non-Restricted:*
   there is a subsequence of σ of length k whose canonization is π.
2. *Restricted:*
   there is a subsequence of σ of length k that is order isomorphic to π.

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And 131, *as in* 1213231, *is a copy of 121 in both the restricted and non-restricted sense.*

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Example

Let $\sigma = 1213231$ *and* $\pi = 121$.

Then 212*, as in* 1213231 *is a copy of 121 in the non-restricted sense, but not in the restricted sense.*

And 131*, as in* 1213231*, is a copy of 121 in both the restricted and non-restricted sense.*

The non-restricted pattern containment definition allows for the blocks of the copy to be in a different order than the blocks in the pattern. The restricted definition does not.

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d(E,k,n) &= \frac{\mu(E,k,n)}{\binom{n}{m}}
\end{aligned}$$

Density

Theorem

For $n \geq m$ we have that $d(E, k, n-1) \geq d(E, k, n)$ and $d(E, k, n) \geq d(E, k-1, n)$. $\square$

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Definition

Let the packing density of the set $E \subseteq \Pi_m$ be

$$\delta(E) = \lim_{n\to\infty} d(E, n, n) = \lim_{n\to\infty} \lim_{k\to\infty} d(E, k, n).$$

Outline

Previous Results

These are the results that I presented in Florence 2009.

Partition π	111	112	123	121
Restricted Packing Density $\delta_r(\pi)$	1	$2\sqrt{3}-3$	1	$\frac{2\sqrt{3}-3}{2}$
Unrestricted Packing Density $\delta(\pi)$	1	$2\sqrt{3}-3$	1	$\frac{1}{4}$

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$1212\cdots12$ is a maximizer

Goal: Show that $121212\cdots12$ is a maximizer for 121.

Packing 121 into 2 block Partitions

Theorem

The number of copies of 121 in a partition among those with two blocks is maximized by the partition $\underbrace{R2D2 \cdots R2D2}_{n}$

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We prove this by proving a series of Lemmas.

Lemma

Let $\pi \in \Pi_{n,2}$ *have exactly two blocks. Assume that* π *consists of i 1's and j 2's with* $i \geq j$. *Then the partition*

$$\hat{\pi} = \underbrace{11\cdots1}_{\lceil(i-j-1)/2\rceil}\ \underbrace{1212\cdots121}_{2j+1}\ \underbrace{11\cdots1}_{\lceil(i-j-1)/2\rceil}$$

satisfies $\nu(121,\hat{\pi}) \geq \nu(121,\pi)$.

Idea of the Proof

- Suppose that inside of the partition we have a string of length $\ell + 2$ of the form

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- Suppose that there are *a* 1's and *b* 2's preceding the initial 2 and *c* 1's and *d* 2's succeeding the final 2.

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- Swapping the intitial 2 with the first 1 in the run gives us a change of $(b + c + \ell - 1) - (a + d + 1)$ copies.
- Swapping the final 2 with the final 1 in the run gives a change of $(a + d + \ell - 1) - (b + c + 1)$ copies.

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- So we can perform at least one of these without losing copies of 121.
- Repeating this process eventually gives us a middle section of alternating 1's and 2's.

Proof that $12121\cdots12$ is maximal in $\Pi_{n,2}$

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- Changing the last 1 in the run of a 1's at the beginning and the first 1 in the run of a 1's at the end, we lose $2ja - j + 2\binom{j+1}{2}$ copies of 121 and gain $2(a-1)(j+a) + \binom{j+1}{2} + \binom{j+2}{2}$ copies of 121.

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- This is a net gain of $a^2 + (a-1)^2$ copies of 121.
- The case where there are $a-1$ 1's at the end gives a net gain of $2(a-1)^2$ copies of 121.
- Thus the partition $\underbrace{1212\cdots121}_{n}$ is a maximizer in $\Pi_{n,2}$

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- Assume that $\pi \in \Pi_n$ has 3 blocks. The first has a elements the second b elements and the third $n - a - b$ elements. Then the maximal number of copies in π is $g(a + b) + g(n - a) + g(n - b)$. This expression is maximized when $a = b = n/3$. So the number of copies of 121 in π is bounded by $3g(2n/3) = \frac{n^3}{27} - \frac{n}{12}$.

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- A similar argument shows that the maximal number of copies in a partition with k blocks is also less that $g(n)$.
- Finally, $\delta(121) = \lim_{n\to\infty} \frac{g(n)}{\binom{n}{3}} = \frac{1}{4}$.

The End of the Death Star

Death Star Explosion

Ur Oh!

Ur Oh!

Thank you in Wookiee.

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- Christopher Lippay
- Julie Lippay, and
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