q-Fibonacci Numbers and Set Partition Statistics

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June 12, 2007

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Let

$$\Pi_n = \{\pi : \pi \vdash [n]\}.$$

Example

$137/25/46 \vdash [7]$

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Proof: $\#\Phi_0 = 1$ because of the empty partition.
$\#\Phi_1 = 1$, as the only partition in Φ_1 is $\{1\}$
and $\#\Phi_n = \#\Phi_{n-1} + \#\Phi_{n-2}$. The first term counts those partitions in Φ_n whose last block is $\{n\}$. The last term counts those partitions in Φ_n whose last block is $\{n-1, n\}$. $\square$

Set Partition Statistics

Question

How are set partition statistics distributed on Φ_n?

The *rb* Statistic

Definition

Let $\pi = B_1/B_2/\ldots/B_k \in \Pi_n$ *and* $b \in B_i$*, then* (b, B_j) *is called a* *right-bigger pair* *if* $i < j$ *and* $b < \max B_j$.

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$rb(\sigma) = 6$

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Only the statistics rb and ls introduced by Wachs and White appear to have nontrivial distributions on Φ_n.

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Observation

If a partition $\pi = B_1/B_2/\ldots/B_k$ *is layered, then the contribution of* B_i *to* $rb(\pi)$ *is* $\min B_i - 1$*.*

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12/3/4/56/78

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we obtain

$$F_0(x,y,q) = 1,\ F_1(x,y,q) = x,$$

and

$$F_n(x,y,q) = xq^{n-1} F_{n-1}(x,y,q) + yq^{n-2} F_{n-2}(x,y,q).$$

q-Fibonacci Numbers

These are related to q-Fibonacci numbers introduced by Carlitz (1974,1975) and Cigler (2003,2004).

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$$F_n(x, y, q) = \sum_{2k \le n} x^{n-2k} y^k q^{\binom{n}{2} - k(n-k)} \left[\begin{array}{c} n-k \\ k \end{array} \right]_q.$$

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We will provide a bijective proof of this theorem, which uses a bijection with integer partitions.

Preliminaries

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$$\sum_{\pi \in \Phi_n} x^{s(\pi)} y^{d(\pi)} q^{rb(\pi)} = \sum_{2k \leq n} \sum_{\pi \in \Phi_{(n,k)}} x^{s(\pi)} y^{d(\pi)} q^{rb(\pi)}$$

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Showing that

$$\sum_{\pi \in \Phi_{(n,k)}} q^{rb(\pi)} = q^{\binom{n}{2} - k(n-k)} \begin{bmatrix} n-k \\ k \end{bmatrix}_q$$

proves the theorem.

Proof

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Let $\pi \in \Phi_{(n,k)}$ have doubletons $B_{i_1}, B_{i_2}, \ldots, B_{i_k}$.

Observation
$rb(\pi) = \binom{n}{2} - \sum_{j=1}^{k} \min B_{i_j}$.

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Observation

$rb(\pi) = \binom{n}{2} - \sum_{j=1}^{k} \min B_{i_j}$.

Example

Compare the partitions $\tau_1 = 1/2/3/45/6/7$ *and* $\tau_2 = 1/2/3/4/5/6/7$*.*

Proof

Thus,

$$\sum_{\pi \in \Phi_{(n,k)}} q^{rb(\pi)} = \sum q^{\binom{n}{2} - \sum_{j=1}^{k} \min B_{i_j}},$$

where the sum is over all possible sets of doubletons $B_{i_1}, B_{i_2}, \ldots, B_{i_k}$.

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We may rewrite this as

$$\sum q^{\binom{n}{2} - \sum_{j=1}^{k} \min B_{i_j}} = q^{\binom{n}{2} - kn} \sum q^{\sum_{j=1}^{k} n - \min B_{i_j}}.$$

Proof

Definition

An integer partition of the integer n is a weakly decreasing sequence of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $|\lambda| \doteq \sum_{i=1}^{k} \lambda_i = n$; the λ_i are called parts.

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We will provide a bijection between $\Phi_{(n,k)}$ and E_{n-1}^{k} and use the fact above to finish the proof.

Proof

Recall that we have

$$\sum_{\pi \in \Phi_{(n,k)}} q^{rb(\pi)} = q^{\binom{n}{2} - kn} \sum q^{\sum_{j=1}^{k} n - \min B_{i_j}}.$$

Proof

Recall that we have

$$\sum_{\pi \in \Phi_{(n,k)}} q^{rb(\pi)} = q^{\binom{n}{2}-kn} \sum q^{\sum_{j=1}^{k} n-\min B_{i_j}}.$$

Consider the map $\psi : \Phi_{(n,k)} \to E_{n-1}^{k}$, which takes a partition $\pi \in \Phi_{(n,k)}$ with doubletons $B_{i_1}, B_{i_2}, \ldots, B_{i_k}$ to the integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where $\lambda_j = n - \min B_{i_j}$.

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Consider the map $\psi : \Phi_{(n,k)} \to E^k_{n-1}$, which takes a partition $\pi \in \Phi_{(n,k)}$ with doubletons $B_{i_1}, B_{i_2}, \ldots, B_{i_k}$ to the integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where $\lambda_j = n - \min B_{i_j}$.

Observation

ψ is a bijection, and $|\psi(\pi)| = \sum_{j=1}^{k} n - \min B_{i_j}$.

Proof

Thus we have

$$q^{\binom{n}{2}-kn}\sum q^{\sum_{j=1}^{k} n-\min B_{i_j}} = q^{\binom{n}{2}-kn}\sum_{\lambda\in E_{n-1}^{k}} q^{|\lambda|}$$

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$$= q^{\binom{n}{2}-k(n-k)}\left[\begin{array}{c} n-k \\ k \end{array}\right]_q .$$

This completes the proof. □

Question

There are similar pattern restricted sets of permutations counted by the Fibonacci Numbers. There are many statistics on permutations. What happens if we consider the distribution of certain statistics on that set? (Currently under investigation)

Table of q-Fibonacci Identities

List of q-Fibonacci Identites
$F_{n+2}(x,y,q) = x^{n+2}q^{\binom{n+2}{2}} + \sum_{j=0}^{n} x^j y q^{\binom{j}{2}} q^j F_{n-j}(xq^{j+2}, yq^{j+2}, q)$
$F_{2n+1}(x,y,q) = \sum_{j=0}^{n} q^{j(j+1)} F_{2n-2j}(xq^{2j+1}, yq^{2j+1}, q)$
$F_{2n}(x,y,q) =$ $y^n q^{n(n-1)} + \sum_{j=0}^{n-1} xy^j q^{j(j-1)} F_{2(n-j)-1}(xq^{2j+1}, yq^{2j+1}, q)$
$F_{m+n}(x,y,q) = F_m(x,y,q)F_n(xq^m, yq^m, q) + yq^m F_{m-1}(x,y,q)F_{n-1}(xq^{m+1}, yq^{m+1}, q)$
$F_{2n+1}(x,y,q) = \frac{1}{xq^n}\left(F_{n+1}(x,y,q)F_{n+1}(xq^n, yq^n, q) - y^2q^{2n+1}F_{n-1}(x,y,q)F_{n-1}(xq^{n+2}, yq^{n+2}, q)\right)$

Table of q-Fibonacci Identities

List of q-Fibonacci Identites (cont'd)
$F_n(x,y)F_{n+1}(x,y,q) =$ $\sum_{j=0}^{\lfloor n/2\rfloor} xy^{2j}q^{\lfloor (2j)^2/2\rfloor}F_{n-2j}(xq^{2j},yq^{2j},q)F_{n-2j}(xq^{2j+1},yq^{2j+1},q)$ $+\ \sum_{j=0}^{\lfloor n/2\rfloor} xy^{2j}q^{\lfloor (2j+1)^2/2\rfloor}F_{n-2j}(xq^{2j},yq^{2j},q)F_{n-2j}(xq^{2j+1},yq^{2j+1},q)$
$F_{2n-1}(x,y,q) =$ $\sum_{i=1}^{n-1}\Big(xy^{2i}q^{2i^2}F_{2n-2i-2}(xq^{2i+2},yq^{2i+2},q)F_{2n-2i-1}(xq^{2i},yq^{2i},q)$ $+\ xy^{2i+1}q^{2i(i+1)}F_{2n-2i-3}(xq^{2i+2},yq^{2i+2},q)F_{2n-2i-2}(xq^{2i+1},yq^{2i+1},q)\Big)$

Table of q-Fibonacci Identites

List of q-Fibonacci Identites (cont'd)

$$F_{3k-1}(x,y,q) = F_k(x,y,q)F_k(xq^k,yq^k,q)F_{k-1}(xq^{2k},yq^{2k},q)$$

$$+\, yq^{k-1}F_{k-1}(x,y,q)F_{k-1}(xq^{k+1},yq^{k+1},q)F_{k-1}(xq^{2k},yq^{2k},q)$$

$$+\, yq^{2k-1}F_k(x,y,q)F_{k-1}(xq^k,yq^k,q)F_{k-2}(xq^{2k+1},yq^{2k+1},q)$$

$$+\, y^2q^{3k-2}F_{k-1}(x,y,q)F_{k-2}(xq^{k+1},yq^{k+1},q)F_{k-2}(xq^{2k+1},yq^{2k+1},q)$$

$$F_{2n+1}(x,y,q) = \sum_{i,j} x^{2n-2i-2j+1}y^{i+j}q^{n(n-i-j-1)+i+j}(q^{n-j+i+1})^{n-i}\begin{bmatrix} n-j \\ i \end{bmatrix}_q \begin{bmatrix} n-i \\ j \end{bmatrix}$$

Thank You

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