Companion Notes for MATH 229 Sections 12.1 and 12.2.

1. **Answer the following questions for the functions**

$$f(x) = x^4 - 8x^2 - 12.$$

(a) Find all critical points of this function.

Critical points of $f(x)$ occur when $f'(x) = 0$ or when $f'(x)$ is discontinuous. First we find $f'(x)$.

$$f'(x) = 4x^3 - 16x$$

Now, we set $f'(x) = 0$ and we get

$$\begin{aligned} 4x^3 - 16x &= 0 \\ 4x(x^2 - 4) &= 0 \\ 4x(x-2)(x+2) &= 0 \end{aligned}$$

This gives us solutions $x = 0, -2$ or 2.

We also check to see where $f'(x)$ is discontinuous. Notice that we can plug anything in for x, so $f'(x)$ is continuous everywhere.

Thus, our only critical values for x are $0, -2$ and 2. If we plug these into the function we get

$$\begin{aligned} f(0) &= -12 \\ f(-2) &= -28 \\ f(2) &= -28 \end{aligned}$$

So our critical points are $(0, -12)$, $(-2, -28)$ and $(2, -28)$.

(b) Determine the intervals on which the function is increasing or decreasing.

The critical values for x are $0, -2$, and 2. These divide the real line into 4 intervals. They are $(-\infty, -2), (-2, 0), (0, 2)$, and $(2, \infty)$. We test the first derivative on these intervals to tell whether they are increasing or decreasing. If the first derivative is positive on the interval then the function is increasing. If the first derivative is negative on the interval then the function is decreasing.

	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
c	-3	-1	1	3
$f'(c)$	-60	12	-12	60
sign	$-$	$+$	$-$	$+$

Thus, $f(x)$ is decreasing on the intervals $(-\infty, -2)$ and $(0, 2)$ and $f(x)$ is increasing on the intervals $(-2, 0)$ and $(2, \infty)$.

(c) Find all relative extrema of this function and determine whether they are maximums or minimums.

Relative Extrema occur at critical points of a function. So the possible relative extrema are at $0, -2$, and 2. We need only determine if they are maximums, minimums, or neither.

Looking back at our chart we can see that as we move across -2 the sign of $f'(x)$ changes from negative to positive. This means that $(-2, -28)$ is a minimum. As we move across 0 the sign of $f'(x)$ changes from positive to negative. This means that $(0, -12)$ is a maximum. As we move across 2 the sign of $f'(x)$ changes from negative to positive. This means that $(2, -28)$ is a minimum.

(d) Find all points of inflection of this function.

To determine points of inflection we need to find out where $f''(x) = 0$ or where $f''(x)$ is discontinuous. Let's find $f''(x)$ first.

$$f''(x) = 12x^2 - 16$$

Notice like above that $f''(x)$ is continous everywhere. We need only find out when its 0.

$$\begin{aligned} 12x^2 - 16 &= 0 \\ 12x^2 &= 16 \\ x^2 &= \frac{4}{3} \\ x &= \frac{\pm 2\sqrt{3}}{3} \end{aligned}$$

Thus, we have two possible points of inflection $\frac{-2\sqrt{3}}{3}$ or $\frac{2\sqrt{3}}{3}$. This gives us 3 intervals. We test the intervals to see where concavity changes.

	$(-\infty, \frac{-2\sqrt{3}}{3})$	$(\frac{-2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3})$	$(\frac{2\sqrt{3}}{3}, \infty)$
c	-2	0	2
$f''(c)$	32	-16	32
sign	$+$	$-$	$+$

We notice that $f(x)$ is concave upward on the intervals $(-\infty, \frac{-2\sqrt{3}}{3})$ and $(\frac{2\sqrt{3}}{3}, \infty)$ and $f(x)$ is concave downward on the interval $(\frac{-2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3})$.

Since the concavity changes as we move across the points $\frac{-2\sqrt{3}}{3}$ and $\frac{2\sqrt{3}}{3}$, both are points of inflection.

(e) Determine the intervals where the function is concave upward and concave downward.

See the question above.

(f) Use the second derivative to determine whether the relative extrema are relative maximums or relative minimums.

The second derivative test says that we plug our critical values for x into $f''(x)$ and check the sign.

$$\begin{aligned} f''(-2) &= 32 \\ f''(0) &= -16 \\ f''(2) &= 32 \end{aligned}$$

Since $f''(-2) > 0$ this means that $f(x)$ has a relative minimum at -2. Since $f''(0) < 0$ this means that $f(x)$ has a relative maximum at 0. Since $f''(2) > 0$ this means that $f(x)$ has a relative minimum at 2.

2. **Answer the following questions for the function**

$$g(x) = x^{2/3}(x^2 - 1).$$

(a) Find all critical points of this function.

$$g'(x) = \frac{8x^2 - 2}{3x^{1/3}}$$

This means we have critical points $\frac{1}{2}$, 0, and $-\frac{1}{2}$. Notice, $g'(x) = 0$ when $x = \frac{1}{2}$, or $x = -\frac{1}{2}$, and $g'(x)$ is undefined when $x = 0$.

(b) Determine the intervals on which the function is increasing or decreasing.

$g(x)$ is decreasing on the intervals $(-\infty, -\frac{1}{2})$ and $(0, \frac{1}{2})$ and $g(x)$ is increasing on the intervals $(-\frac{1}{2}, 0)$ and $(\frac{1}{2}, \infty)$.

(c) Find all relative extrema of this function and determine whether they are maximums or minimums.

There are minimums at $-\frac{1}{2}$ and $\frac{1}{2}$ and a maximum at 0.

(d) Find all points of inflection of this function.

$$g''(x) = \frac{40x^2 + 2}{9x^{4/3}}$$

The only possible point of inflection is 0, where $g''(x)$ is discontinous. The equation

$$\frac{40x^2 + 2}{9x^{4/3}} = 0$$

has no solutions.

If you test your intervals, you'll find that $g(x)$ is concave upward on both $(-\infty, 0)$ and $(0, \infty)$. Thus, 0 is not an inflection point, because concavity doesn't change.

(e) Determine the intervals where the function is concave upward and concave downward.

See the above answer.

(f) Use the second derivative to determine whether the relative extrema are relative maximums or relative minimums.

You are only able to use the second derivative test on the points $\frac{1}{2}$ and $-\frac{1}{2}$. You can't test 0 this way because $g''(x)$ is undefined at 0.