MATH 261
Exam 3
Review Sheet

This review sheet is intended to remind you of the concepts that you are expected to understand for the exam. It is by no means a complete representation of what could be on the exam. You are responsible for everything discussed in the notes, on labs and in the suggested homework exercises.

1. Use the graph below to answer the following questions.

   (a) Suppose the graph below is a graph of f. State the intervals where f is increasing, decreasing, concave up and concave down. State the locations of the local maxima and local minima and inflection points.

   Solution: Increasing: $[-5, 3]$, $[7, \infty)$, Decreasing: $(-\infty, -5]$, $[3, 7]$, Concave up: $(-\infty, -3]$, $[0, 2.5]$, $[4, \infty)$, Concave down: $[-3, 0]$, $[2.5, 4,]$, Max at $x = 3.$, Min at $x = -5$ and 7, Inflection points at $x = -3, 0, 2.5$ and 4.

   (b) Suppose the graph below is a graph of f'. State the intervals where f is increasing, decreasing, concave up and concave down. State the locations of the local maxima and local minima and inflection points.

   Solution: f is increasing where f' is positive, so f is increasing on $(-\infty, -8]$, $[-3, 5]$ and $[9, \infty)$. f is decreasing where f' is negative so f is decreasing on $[-8, -3]$ and $[5, 9]$. f has a maximum where f' changes from positive to negative and a minimum where f' changes from negative to positive, so f has a maximum at $x = -8$ and 5 and a minimum at $x = -3$ and 9. f is concave up where the f' is increasing and concave down where f' is decreasing, so f is concave up on $[-5, 3]$, $[7, \infty)$ and concave down on $(-\infty, -5]$, $[3, 7]$ and inflection points are where $x = -5, 3$ and 7.

   (c) Suppose the graph below is a graph of f''. State the intervals where f is concave up and concave down. State the locations of any inflection points.

   Solution: f is concave up where f'' is positive and concave down where f'' is negative. So f is concave up on $(-\infty, -8]$, $[-3, 5]$ and $[9, \infty)$ and concave down on $[-8, -3]$ and $[5, 9]$. The inflection points are where $x = -8, -3, 5$ and 9.

2. Suppose the previous graph is a graph of $g(x)$. Answer the following questions.

   (a) Where is $g'(x) = 0$?

   Solution: $g'(x)$ is 0 where the tangent line is horizonatal, so where $x = -5, 0, 3$ and 7

   (b) Where is $g''(x) = 0$?

   Solution: $g''(x)$ is 0 where the concavity changes, so where $x = -3, 0, 2.5$ and 4.

3. Let $f(x) = x^4 - 27x + 22$. Show that $f(x)$ satisfies the hypotheses of Rolle's theorem on $[0, 3]$ and find a number c that satisfies the conclusion of Rolle's Theorem.

   Solution: The hypotheses of Rolle's Theorem say that f must be continuous on $[a, b]$, differentiable on (a, b) and that $f(a) = f(b)$. If this happens then there is some number c in (a, b) with $f'(c) = 0$.

   Since f is a polynomial it is continuous on $[0, 3]$. It's derivative is $f'(x) = 4x^3 - 27$, which is also a polynomial, so f is differentiable on $(0, 3)$. Also, $f(0) = 22$ and $f(3) = 22$. So f satisfies the hypotheses of Rolle's Theorem on $[0, 3]$. Thus there must be some c in $(0, 3)$ with $f'(c) = 0$. Setting $f'(x) = 0$ and solving for x gives us that $c = \frac{3\sqrt[3]{2}}{2}$.

4. Let $g(x) = \frac{x-3}{2x+1}$. Show that $g(x)$ satisfies the hypotheses of the Mean Value Theorem on $[0, 2]$ and find a number c that satisfies the conclusion of the MVT.

   Solution: The hypotheses of the MVT say that g must be continuous on $[a, b]$ and differentiable on (a, b). If this happens then there is some c in (a, b) with $g'(c) = \dfrac{g(b) - g(a)}{b - a}$.

   g has a discontinuity at $-1/2$, but is continuous everywhere else, so it's continuous on $[0, 2]$. $g'(x) = \frac{7}{(2x+1)^2}$, which only has a discontinuity at $x = -1/2$, so g is differentiable on $(0, 2)$. Thus g satisfies the hypotheses of the MVT on $[0, 2]$. Thus, there must be some c in $(0, 2)$ with $g'(c) = \frac{g(b)-g(a)}{b-a}$. That is there must be some c in $(0, 2)$ such that

$$\begin{aligned} \frac{7}{(2c+1)^2} &= \frac{g(2) - g(0)}{2 - 0} \\ \frac{7}{(2c+1)^2} &= \frac{7}{5} \\ 5 &= (2c+1)^2 \\ \frac{-1 \pm \sqrt{5}}{2} &= c \end{aligned}$$

   So $c = \frac{-1+\sqrt{5}}{2}$

5. Suppose a Vespa is driving along a straight highway in North Dakota for 60 seconds. Every 10 seconds the position, velocity and acceleration are measured. Distance is measured in feet and time is measured in seconds. The data are in the following chart. Use the chart to answer the following questions. Be sure to state the theorem that you are using to support your answer.

time	0	10	20	30	40	50	60
$s(t)$	0	20	50	90	130	190	210
$v(t)$	0	5	5	5.5	4	3	0
$a(t)$	1	5	7	10	2	2	1

(a) Is there a time when the Vespa is 75 feet from the start?

Solution: Yes. The position of the Vespa is a continuous function of time, and the Vespa goes from 50 feet to 90 feet from the start during the 20 to 30 second time interval. So by the Intermediate Value Theorem the Vespa must have been 75 miles from the start at some time between 20 and 30 seconds.

(b) Is there a time when the Vespa is traveling 6 feet per second?

Solution: Yes. Using the MVT, $s(t)$ is continous on $[40, 50]$ and differentiable on $(40, 50)$, so there must be a time t in $(40, 50)$ when

$$s'(t) = v(t) = \frac{s(50) - s(40)}{50 - 40} = \frac{60}{10} = 6.$$

(c) Is there a time when the acceleration of the Vespa is 0?

Solution: Yes. Using Rolle's Theorem, we note that $v(t)$ is continuous on $[10, 20]$ and differentiable on $(10, 20)$ and $v(10) = 5 = v(20)$. Thus, there must be a time between 10 and 20 seconds when $v'(t) = a(t) = 0$.

6. Find the absolute extrema of $f(x) = \frac{x^2}{x^2+4}$ on $[-1, 3]$.

To find the absolute extrema of $f(x)$ on $[-1, 3]$, we first note that f is continuous on $[-1, 3]$, since it is continuous everywhere (the denominator is always positive). Now, we find any critical numbers on $(-1, 3)$. $f'(x) = \dfrac{8x}{(x^2 + 4)^2}$, so $f'(x)$ exists everywhere and is 0 only when $x = 0$. Our only critical number is 0 and 0 is in $(-1, 3)$. Now, we just check the value of $f(x)$ at $-1, 0$ and 3.

$f(-1) = 1/5$
$f(0) = 0$
$f(3) = 9/13$
.

Thus, the absolute maximum of $f(x)$ on $[-1, 3]$ is $f(3) = 9/13$ and the absolute minimum is $f(0) = 0$.

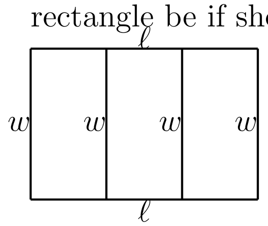
7. Find the absolute extrema of $g(x) = x^3 - 4x^2 + 5x - 12$ on $[0, 3]$.

Note that since $g(x)$ is a polynomial, g is continuous on $[0, 3]$ and differentiable on $(0, 3)$. Now, we find any critical numbers on $(0, 3)$. $g'(x) = 3x^2 - 8x + 5$, so $g'(x)$ exists everywhere and is 0 when $x = 1$ or $5/3$. Both of these critical numbers are in $(0, 3)$. Now, we just check the value of g when $x = 0, 1, 5/3$ and 3.

$g(0) = -12$
$g(1) = -10$
$g(5/3) = -274/27 \approx -10.15$
$g(3) = -6$

Thus, the absolute maximum of $g(x)$ on $[0, 3]$ is $g(3) = -6$ and the absolute minimum is $g(5/3) \approx -10.15$.

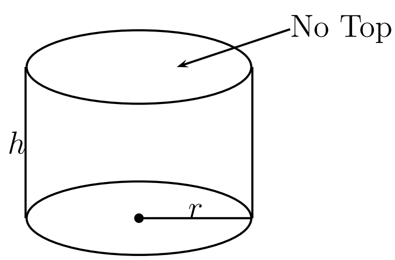
8. A farmer has 4000 feet of fence and wants to fence off a rectangular plot of land for her livestock that looks like the picture below. What should the dimensions of the rectangle be if she wants to maximize area?



Solution: We want to maximize the area. The area is given by $A = \ell w$. We want A to be a function of one variable. Notice that because she only has 4000 feet of fence we have to have that $2\ell + 4w = 4000$. Solving this equation for ℓ gives us that $\ell = 2000 - 2w$. So we have a function $A(w) = (2000 - 2w)w$. Now, we distribute the w through and take the derivative. $A'(w) = 2000 - 4w$. $A'(w)$ is continous everywhere, so we set this equal to 0 solve for w and get $w = 500$. Now, we need to be sure that this is a maximum. $A''(w) = -4 < 0$, so $w = 500$ gives us a maximum. The dimensions that maximize the area are 500 feet by 1000 feet.

9. A cylindrical container without a top is to be constructed from tin. The volume of the container needs to be 8 cubic inches. If there is no waste in construction, find the dimensions of the container that requires the least amount of material. (Hint: minimize surface area)

Solution:



We want to minimize surface area. Using the picture we can see that surface area is the area of the bottom of the cylinder plus the area of the sides. This is $A = \pi r^2 + 2\pi rh$. We also know that the volume has to be 8 cubic inches, so $\pi r^2 h = 8$. Solving this equation for h gives us $h = \frac{8}{\pi r^2}$. If we plug this into the surface area equation then we get $A(r) = \pi r^2 + 2\pi r(\frac{8}{\pi r^2})$. Now, we simplify and take the derivative and obtain $A'(r) = 2\pi r - \frac{16}{r^2}$. $A'(r)$ is discontinous when $r = 0$, but this case isn't interesting because then we wouldn't even have a cylinder. When we set $A'(r) = 0$ and solve for r, we get $r = \frac{2}{\sqrt[3]{\pi}} \approx 1.37$ inches. We must check that this gives us a minimum. $A''(r) = 2\pi + \frac{32}{r^3}$, which is positive when $r \approx 1.37$, so it does give a minimum. Thus, the dimensions that minimize the surface area are $r \approx 1.37$ and $h \approx 1.37$.

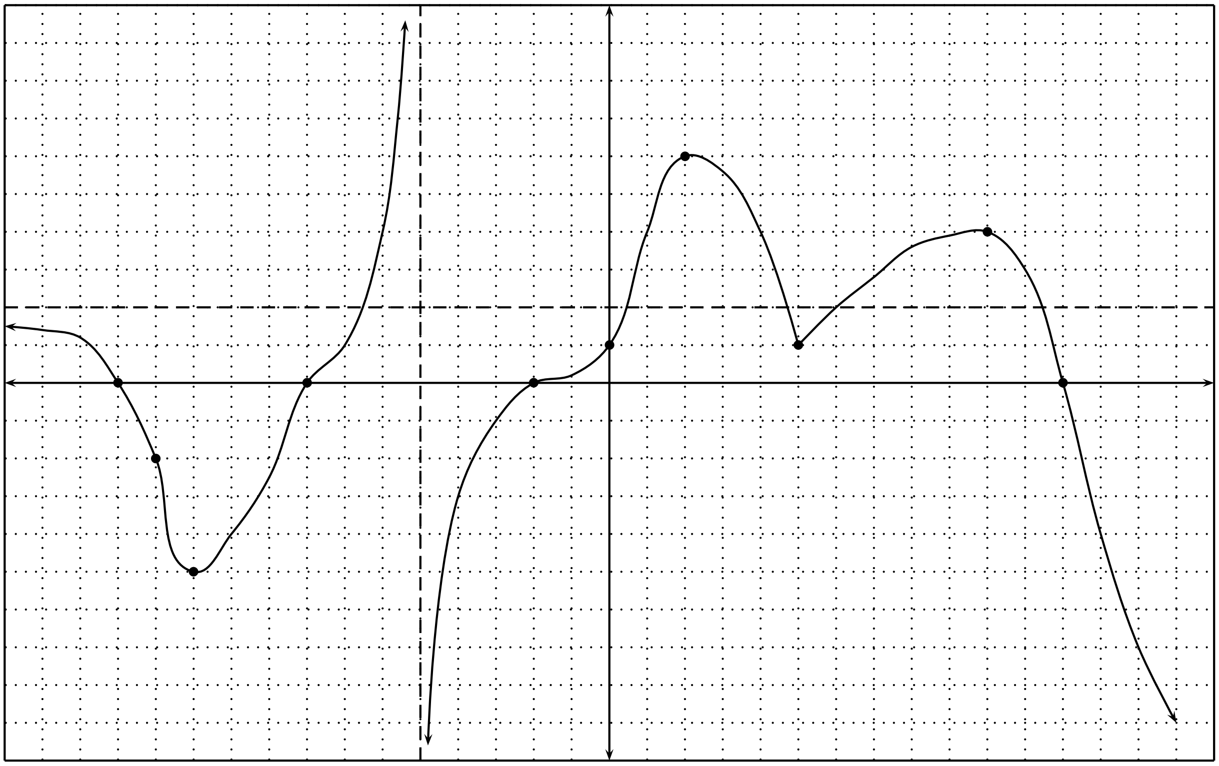
10. A company that conducts bus tours found that when the price was \$9 per person, the average number of customers was 1000 per week. When the company reduced the price to \$7 per person, the average number of customers increased to 1500 per week. Assuming that the demand function is linear, what price should be charged to obtain the maximum weekly revenue?

    Solution: We put together a demand equation to start. We have two points on our line $(1000, 9)$ and $(1500, 7)$. This gives us slope $\frac{-1}{250}$. We get that $p(x) = \frac{-1}{250}x + 13$. Thus, the revenue is $R(x) = xp(x) = \frac{-1}{250}x^2 + 13x$. We want to maximize this, so we take the derivative and get $R'(x) = \frac{-1}{125} + 13$. $R'(x)$ exists everywhere, so we just look for when it's 0. If you set $R'(x) = 0$ and solve for x you get $x = 1625$. We want to know what price to charge, so we plug this back into $p(x)$ and get \$6.50. If they charge \$6.50 they will maximize revenue.

11. A bulk grocer charges \$0.35 per pound for flour if you buy 10 pounds of flour. The grocer will reduce the price by \$0.02 per pound each time you buy 5 more pounds of flour. At what price is the grocer's revenue on flour the highest?

    Solution: We can do this problem just like the problem above. We have two points on our price equation $(10, .35)$ and $(15, .33)$. Using these two points we get that slope is -0.004, and we get equation $p(x) = -0.004x + 0.39$. Thus, the revenue function is $R(x) = -0.004x^2 + 0.39x$. We take the derivative and get $R'(x) = -0.008x + 0.39$. $R'(x)$ exists everywhere, so we just set $R'(x) = 0$ and solve for x. We get $x = 48.75$, this gives us price should be \$0.20.

12. Use the grid below to graph a function $f(x)$ that has the following properties.
    Domain: $(-\infty, -5) \cup (-5, \infty)$
    x-intercepts: $(-13, 0)$, $(-8, 0)$, $(-2, 0)$ $(12, 0)$
    y-intercept: $(0, 1)$ Increasing on $(-11, -5)$, $(-5, 2)$, and $(5, 10)$
    Decreasing on $(-\infty, -11)$, $(2, 5)$, and $(10, \infty)$
    Concave up on $(-12, -5)$, $(-2, 0)$, and $(12, \infty)$
    Concave down on $(-\infty, -12)$, $(-5, -2)$, $(0, 5)$, and $(5, 12)$
    Local Max: $(2, 6)$ and $(10, 4)$
    Local Min: $(-11, -5)$ and $(5, 1)$
    Inflection points $(-12, -2)$, $(-2, 0)$, $(0, 1)$ and $(12, 0)$
    $\lim_{x\to-\infty} f(x) = 2$, $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-5^-} f(x) = \infty$, $\lim_{x\to-5^+} f(x) = -\infty$



13. For each of the following functions, determine the domain of the function, the x and y intercepts, where the function is increasing and decreasing, any local extrema, where the function is concave up and concave down, find any inflection points and find any asymptotes. Use this information to sketch a graph of the function.

   (a) $f(x) = x^4 - 12x^2$

   Solution: Domain: All real numbers.
   x-intercepts: $0 = x^4 - 12x^2$ gives $x = 0, \pm 2\sqrt{3}$. This gives $(0,0)$, $(2\sqrt{3},0)$, $(-2\sqrt{3},0)$.
   y-intercepts: $f(0) = 0^4 - 12(0)^2 = 0$. This gives $(0,0)$. Increasing and Decreasing. $f'(x) = 4x^3 - 24x$. $f'(x)$ exists everywhere. Set $f'(x) = 0$. We get $x = 0, \pm\sqrt{6}$.

	$(-\infty,-\sqrt{6})$	$(-\sqrt{6},0)$	$(0,\sqrt{6})$	$(\sqrt{6},\infty)$
c	-3	-1	1	3
$f'(c)$	-36	20	-20	36
sign	-	+	-	+

   So $f(x)$ is increasing on $(-\sqrt{6},0)$ and $(\sqrt{6},\infty)$ and decreasing on $(-\infty,-\sqrt{6})$ and $(0,\sqrt{6})$.
   Local extrema. $(-\sqrt{6},-36)$ and $(\sqrt{6},-36)$ are minima and $(0,0)$ is a maximum.
   Concavity. $f''(x) = 12x^2 - 24$. $f''(x)$ exists everywhere. Set $f''(x) = 0$. We get $x = \pm\sqrt{2}$.

	$(-\infty,-\sqrt{2})$	$(-\sqrt{2},\sqrt{2})$	$(\sqrt{2},\infty)$
c	-2	0	2
$f''(c)$	24	-24	24
sign	+	-	+

So $f(x)$ is concave up on $(-\infty,-\sqrt{2})$ and $(\sqrt{2},\infty)$ and concave down on $(-\sqrt{2},\sqrt{2})$.

Inflection points. The inflection points are $(-\sqrt{2},-20)$ and $(\sqrt{2},-20)$.

Asymptotes: Since there are no x's in the denominator of this function, there are no vertical asymptotes. Also, $\lim_{x\to\infty} f(x)=\infty$ and $\lim_{x\to-\infty} f(x)=\infty$, so there are no horizontal asymptotes.

(b) $g(x)=\frac{x+2}{x-3}$

Solution:

Doman: $(-\infty,3)\cup(3,\infty)$.
x-intercepts: $(-2,0)$
y-intercept:$(0,-2/3)$
Increasing and Decreasing. $g'(x)=\frac{-5}{(x-3)^2}$. This is undefined at $x=3$. Setting $g'(x)=0$ and solving for x gives no solution. So we have no critical numbers.

	$(-\infty,3)$	$(3,\infty)$
c	0	4
$f'(c)$	-5/9	-5
sign	-	-

So $g(x)$ is decreasing on $(-\infty,3)$ and on $(3,\infty)$.

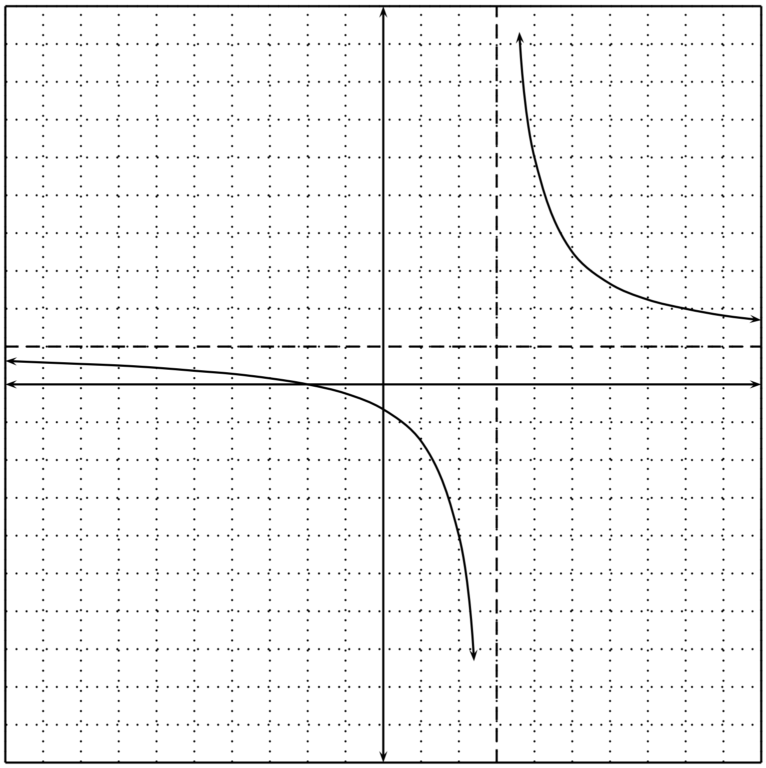
Since there are no critical points there are no local extrema.

Concavity. $g''(x)=\frac{10}{(x-3)^3}$, which is undefined at $x=3$. Setting $g''(x)=0$ and solving for x gives no solution. So we have no inflection points.

	$(-\infty,3)$	$(3,\infty)$
c	0	4
$f'(c)$	-10/27	10
sign	-	+

So $g(x)$ is concave down on $(-\infty,3)$ and concave up on $(3,\infty)$.

Asymptotes: Since $x-3$ is in the denominator of $g(x)$ we have a vertical asymptote of $x=3$. And $\lim_{x\to\infty} g(x)=1$ and $\lim_{x\to-\infty} g(x)=1$, so we have a horizontal asymptote of $y=1$.



14. The position function of a particle traveling in a straight line is given by $s(t) = -2t^3 - 3t^2 + t - 14$. Find the velocity of the particle at time t. Find the acceleration of the particle at time t. Find the position, velocity and acceleration after 2 seconds. Find the time on the interval $[0, 4]$ when speed is greatest. (Remember speed is $|v(t)|$.)

    Solution: The velocity at time t is $v(t) = s'(t) = -6t^2 - 6t + 1$ and the acceleration at time t is $a(t) = v'(t) = -12t - 6$. The position at $t = 2$ is $s(2) = -40$ feet, the velocity at $t = 2$ is $v(2) = -110$ feet per second, and the acceleration at time $t = 2$ is $a(t) = -30$ feet per second per second. We want to maximize $|v(t)|$ on the interval $[0, 4]$. First, we find any critical numbers of $v(t)$ on $[0, 4]$. Since $a(t)$ is continuous everywhere we set $a(t) = 0$ and obtain $t = 1/2$. We get that $v(0)$=1, $v(1/2) = -3.5$ and $v(4) = -119$. So the maximum speed is 119 ft/sec.

15. Use Newton's Method to approximate $\sqrt{7}$ to 4 decimal places.

    Solution: See solution on page 233 in your book

16. Use Newton's Method to approximate the real root of $x^3 + 2x - 7$ to 5 decimal places. Use $x_1 = 2$.

    Solution: Let $f(x) = x^3 + 2x - 7$. Recall that $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. We have $f'(x) = 3x^2 + 2$. We'll choose $x_1 = 2$. Then

    $x_2 = 1.64286$
    $x_3 = 1.57157$

$x_4 = 1.56895$
$x_5 = 1.56895$

So the real root of $x^3 + 2x - 7$ is approximately 1.56895.