Math 261
Curve Sketching Handout
10/25/2007

A Guide to Curve Sketching

A. Finding Intercepts

In order to understand the graph of a function $f(x)$, knowing where the graph crosses the x and y axes can be quite useful.

1. To find the y-intercept, we simply compute $f(0)$, if it exists.
2. To find the x-intercepts, we find solutions to the equation $f(x) = 0$. Note: finding solutions to this equation using algebra may not be easy to do. If $f(x)$ factors, or has an algebraic form that is fairly simple, then we can find the intercepts. Otherwise, we can use other means to estimate the x-intercepts.

B. Finding Increasing/Decreasing Intervals and Relative Extrema Using $f'(x)$

Remember that $f(x)$ is increasing whenever $f'(x) > 0$, and $f(x)$ is decreasing whenever $f'(x) < 0$. Therefore, we will analyze the first derivative of $f(x)$ in order to understand its increasing/decreasing behavior. To do this, we use the following procedure:

1. Given $f(x)$, compute the derivative $f'(x)$
2. Find all critical points. Remember, there are two types:
   (a) Find all x for which $f'(x)$ is *undefined*. To do this, we need to understand the domain of $f'(x)$.
   (b) Find all x for which $f'(x) = 0$. To do this, we use algebraic methods to solve this equation - often factoring.
3. Analyze the sign of $f'(x)$ by making a sign testing diagram and choosing appropriate test values. Evaluate $f'(x)$ on each test value, and complete the sign diagram for $f'(x)$.
4. Use the sign diagram to read off the intervals where $f(x)$ is increasing (that is, where $f'(x) > 0$) and the intervals where $f(x)$ is decreasing (that is, where $f'(x) < 0$).
5. Use the sign diagram to find relative extrema. Points at which $f(x)$ is defined, and $f'(x)$ goes from positive to negative are relative maxima. Points at which $f(x)$ is defined, and $f'(x)$ goes from negative to positive are relative minima.

Note: It is a good idea to go ahead and find the y values for each of the relative extrema you find. To do this, we use the *original function* $f(x)$.

C. Finding Conavity and Inflection Points Using $f''(x)$

Remember that $f(x)$ is concave up whenever $f''(x) > 0$, and $f(x)$ is concave down whenever $f''(x) < 0$. Therefore, we will analyze the *second* derivative of $f(x)$ in order to understand its concavity. To do this, we use the following procedure:

1. Given $f(x)$, compute the derivative $f'(x)$, and then the second derivative $f''(x)$.
2. Find the key values for the second derivative. There will once again be two types:
   (a) Find all x for which $f''(x)$ is *undefined*. To do this, we need to understand the domain of $f''(x)$.
   (b) Find all x for which $f''(x) = 0$. To do this, we use once again use algebraic methods to solve this equation.
3. Analyze the sign of $f''(x)$ by making a sign testing diagram and choosing appropriate test values. Evaluate $f''(x)$ on each test value, and complete the sign diagram for $f''(x)$.
4. Use the sign diagram to read off the intervals where $f(x)$ is concave up (that is, where $f''(x) > 0$) and the intervals where $f(x)$ is concave down (that is, where $f''(x) < 0$).
5. Use the sign diagram to find all inflecion points. Points at which $f(x)$ is defined, and $f''(x)$ goes from positive to negative **or** from negative to positive are inflection points.

   Note: It is a good idea to go ahead and find the y values for each of the inflection points you find. To do this, we use the *original function* $f(x)$.

D. Finding Asymptotes to the graph of $f(x)$

1. To determine whether $f(x)$ has any horizontal asymptotes, we need to look at the following limits:

   $\lim_{x \to -\infty} f(x)$, and $\lim_{x \to \infty} f(x)$

   If either of these limits has a finite value, then the function $f(x)$ has a horizontal asymptote in the corresponding direction. Note that for most of the functions we look at, either there will be no horizontal asymptote, or both infinite limits will agree and we will have a single horizontal asymptote. It is possible to have "one-sided" horizontal asymptotes though.

2. To determine whether $f(x)$ has any vertical asymptotes, we look at the domain of $f(x)$, and we investigate the behavior of the function at each point in the domain where $f(x)$ is undefined to see if it exhibits infinite behavior at that point. To do so, we compute the one-sided limits:

   $\lim_{x \to a^-} f(x)$, and $\lim_{x \to a^+} f(x)$

   at each point $x = a$ where $f(x)$ is undefined. If these limits are $\pm\infty$, then there is a vertical asymptote at $x = a$.

E. Combining All this Information to Skeech the graph of $f(x)$

1. First, draw your coordinate axes, sketch in any vertical and horizontal asymptotes, and plot the points associated with your x-intercepts, y-intercepts, relative extrema, and inflection points.

2. Combine the information from the first and second derivatives to sketch the rest of the graph. To do this, for each interval between extrema and/or inflection points, think about what the concavity is and whether the interval is increasing of decreasing, and draw in an appropriate curve from our four protoypical graph shapes (inceasing/concave up, inceasing/concave down, deceasing/concave up, deceasing/concave down)