Math 261
Exam 2 - Practice Problems
Solutions

1. Find the derivative $y' = \frac{dy}{dx}$ for each of the following:

   (a) $y = e^2 x + ex^2$
   $y' = e^2 + 2ex$

   (b) $y = \cot x$
   $y' = -\csc^2 x$

   (c) $y = \sqrt{x}\sec(x^2)$
   $y' = \frac{1}{2}x^{-\frac{1}{2}}\sec(x^2) + x^{\frac{1}{2}}\sec(x^2)\tan(x^2)\cdot 2x$
   $y' = \dfrac{1}{2\sqrt{x}}\sec(x^2) + 2x\sqrt{x}\sec(x^2)\tan(x^2)$

   (d) $y = 2\tan^3(2x^3)$
   $y' = 6\tan^2(2x^3)\cdot\sec^2(2x^3)\cdot 6x^2 = 36x^2\tan^2(2x^3)\sec^2(2x^3)$

   (e) $y = \dfrac{x^2 - 7\cos(3x)}{x + \sin(3-2x)}$
   $y' = \dfrac{(2x + 21\sin(3x))(x + \sin(3-2x)) - (x^2 - 7\cos(3x))(1 - 2\cos(3-2x))}{(x + \sin(3-2x))^2}$
   Note: I won't make you take time to simplify problems like this one on the exam.

   (f) $x^2 y + 3xy - 5y^2 = 7$
   Differentiating with respect to x: $2xy + x^2y' + 3y + 3xy' - 10yy' = 0$
   Then $(x^2 + 3x - 10y)y' = -2xy - 3y$
   Thus $y' = \dfrac{-2xy - 3y}{x^2 + 3x - 10y}$

   (g) $\cos^2(xy) = 1$
   Differentiating with respect to x: $y' = 2\cos(xy)\cdot(-\sin(xy))\cdot(y + xy') = 0$
   Then $-2y\cos(xy)\sin(xy)) - [2x\cos(xy)\sin(xy))]y' = 0$,
   or $-y'[2x\cos(xy)\sin(xy))] = 2y\cos(xy)\sin(xy)$
   Thus $y' = \dfrac{2y\cos(xy)\sin(xy)}{-2x\cos(xy)\sin(xy)} = -\dfrac{y}{x}$

2. Use the formal limit definition of the derivative to find the derivative of the following:

   (a) $f(x) = x^2 - 3x$
   $f'(x) = \lim_{h\to 0}\dfrac{(x+h)^2 - 3(x+h) - x^2 + 3x}{h} = \lim_{h\to 0}\dfrac{x^2 + 2xh + h^2 - 3x - 3h - x^2 + 3x}{h}$
   $= \lim_{h\to 0}\dfrac{2xh + h^2 - 3h}{h} = \lim_{h\to 0} 2x + h - 3 = 2x - 3$

(b) $f(x) = \dfrac{2}{x-3}$

$$f'(x) = \lim_{h\to 0} \frac{\frac{2}{x+h-3} - \frac{2}{x-3}}{h} = \lim_{h\to 0} \frac{\frac{2(x-3)-2(x+h-3)}{(x+h-3)(x-3)}}{h}$$

$$= \lim_{h\to 0} \frac{2x-6-2x-2h+6}{(x+h-3)(x-3)} \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{-2h}{(x+h-3)(x-3)} \frac{1}{h} = \lim_{h\to 0} \frac{-2}{(x+h-3)(x-3)} = \frac{-2}{(x-3)^2}$$

(c) $f(x) = \sqrt{x-2}$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h-2} - \sqrt{x-2}}{h} \cdot \frac{\sqrt{x+h-2} + \sqrt{x-2}}{\sqrt{x+h-2} + \sqrt{x-2}}$$

$$= \lim_{h\to 0} \frac{x+h-2-x+2}{h(\sqrt{x+h-2} + \sqrt{x-2})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h-2} + \sqrt{x-2})} = \frac{1}{2\sqrt{x-2}}$$

3. Use the quotient rule to derive the formula for the derivative of $\tan(x)$.

   Suppose $f(x) = \tan x = \dfrac{\sin x}{\cos x}$

   Then, using the quotient rule:

   $$f'(x) = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2(x).$$

   Thus $\frac{d}{dx}(\tan x) = \sec^2 x$

4. If $f(x) = \sqrt{3x-5}$, find the intervals where $f(x)$ is continuous, and find the intervals where $f(x)$ is differentiable.

   Recall that $f(x)$ is the square root of a polynomial, so it is continuous wherever it is defined. That is, whenever $3x - 5 \geq 0$, or when $x \geq \frac{5}{3}$.

   Thus $f(x)$ is continuous on $[\frac{5}{3}, \infty)$

   Next, $f'(x) = \dfrac{1}{2}(3x-5)^{-\frac{1}{2}}(3) = \dfrac{3}{2\sqrt{3x-5}}$

   Then $f'(x)$ is defined when $3x - 5 > 0$, or when $x > \frac{5}{3}$, so $f(x)$ is differentiable on $(\frac{5}{3}, \infty)$

5. If $f(x) = 3x^4 - 5x^2 + 7x - 12$, use differentials to approximate $f(1.1)$

   Let $x = 1$ and $\Delta x = .1$. Notice that $f'(x) = 12x^3 - 10x + 7$, so $f'(1) = 12 - 10 + 7 = 9$, and $f(1) = 3 - 5 + 7 - 12 = 10 - 17 = -7$

   Then $f(1.1) \approx f(1) + f'(1)\Delta x = -7 + 9(.1) = -7 + .9 = 6.1$

6. Use differentials to estimate $\sqrt[3]{9}$. How good is your estimate?

   Let $f(x) = x^{\frac{1}{3}}$, $x = 8$, and $\Delta x = 1$.

   Then $f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}$.

   Therefore, $f(8) = \sqrt[3]{8} = 2$, and $f'(8) = \frac{1}{3 \cdot 8^{\frac{2}{3}}} = \frac{1}{3(4)} = \frac{1}{12}$.

   Thus $f(9) \approx f(8) + f'(8)\Delta x = 2 + \frac{1}{12} = \frac{25}{12} \approx 2.08333$

   Notice that $\sqrt[3]{9} \approx 2.08008$, so our approximation in within 33 ten-thousandths.

7. Suppose helium is being pumped into a spherical baloon at a rate of 4 cubic feet per minute. Find the rate at which the radius is changing when the radius is 2 feet.

   Recall that the volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$. In the situation described, both V and r are functions of time t in minutes.

   Then, differentiating implicitly, $\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$.

   We also know that $\dfrac{dV}{dt} = 4\dfrac{ft^3}{min}$ and $r = 2$ feet.

   Thus $4 = 4\pi(4)\dfrac{dr}{dt}$, so $\dfrac{dr}{dt} = \dfrac{4}{16\pi} = \dfrac{1}{4\pi}\dfrac{ft}{min}$.

8. Find the equation of the tangent line to the graph of $f(x) = \tan(4x)$ when $x = \frac{3\pi}{16}$

   $f'(x) = 4\sec^2(4x) = \dfrac{4}{\cos^2(4x)}$. Therefore $f'(\frac{3\pi}{16}) = \dfrac{4}{\cos^2(\frac{3\pi}{16})} = \dfrac{4}{\left(\frac{\sqrt{2}}{2}\right)^2} = \dfrac{4}{\frac{1}{2}} = 8$,

   and $f(\frac{3\pi}{16}) = \tan(\frac{3\pi}{4}) = -1$.

   Then the tangent line to $f(x)$ when $x = \frac{3\pi}{16}$ has slope 8 and goes through the point $\left(\frac{3\pi}{16}, -1\right)$

   Hence the tangent line has equation $y + 1 = 8\left(x - \frac{3\pi}{16}\right)$ so $y = 8x - \frac{3\pi}{2} - 1$

9. Find the points on the graph of $y = 2x^3 + 3x^2 - 72x + 5$ at which the tangent line is horizontal.

   Let $y = f(x)$. Then $f'(x) = 6x^2 + 6x - 72 = 6(x^2 + x - 12)$, so the points at which the tangent line is horizontal occur when $x^2 + x - 12 = 0$, or when $(x+4)(x-3) = 0$, that is, when $x = -4$, and $x - 3$.

   Notice that $f(-4) = 2(-4)^3 + 3(-4)^2 - 72(-4) + 5 = 213$, and $f(3) = 2(3)^3 + 3(3)^2 - 72(3) + 5 = -130$

   Hence the points on the graph of $y = f(x)$ with horizontal tangent lines are: $(-4, 213)$ and $(3, -130)$.

10. Find the equation of the tangent line to the graph of the relation $x^3 + x^2y = \sqrt{y^3} - 3$ at the point $(1, 4)$.

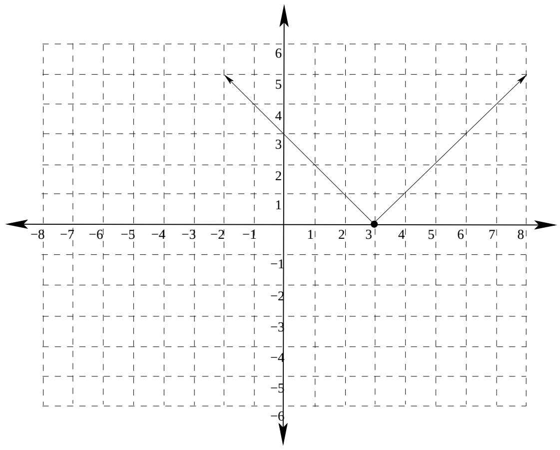
    Differentiating implicitly, $3x^2 + 2xy + x^2y' = \frac{3}{2}y^{\frac{1}{2}}y'$

    Then when $x = 1$ and $y = 4$, $3(1) + 2(1)(4) + (1)y' = \frac{3}{2}(2)y'$, or $11 + y' = 3y'$

    Therefore, $11 = 2y'$ so $y' = \frac{11}{2}$, so the tangent line has equation $y - 4 = \frac{11}{2}(x-1)$, or $y = \frac{11}{2}x - \frac{11}{2} + 4 = \frac{11}{2}x - \frac{3}{2}$.

11. Draw the graph of a function $f(x)$ that is continuous when $x = 3$, but is not differentiable when $x = 3$.

There are many possible examples. One possibility is:



12. Find $g'(2)$ if $h(x) = f(g(x))$, $f(3) = -2$, $g(2) = 3$, $f'(3) = 5$, and $h'(2) = -30$.

Using the Chain Rule, $h'(x) = f'(g(x))g'(x)$, so $h'(2) = f'(g(2))g'(2) = f'(3)g'(2)$.

Therefore, $-30 = 5g'(2)$, so $-6 = g'(2)$.

13. Find $f^{(8)}(x)$ if $f(x) = \sin(2x)$

Notice that $f'(x) = 2\cos(2x)$

Continuing in this fashion, $f^{(8)}(x) = 2^8 \sin(2x) = 256 \sin(2x)$.

14. Find $f^{(13)}(x)$ if $f(x) = x^{12} + 7x^5 - 3x^3 - 1$

Since the highest exponent is 12, and differentiation using the power rule lowers the exponent of each term by one, then $f^{(13)}(x) = 0$.