Math 261
Exam 4 - Practice Problems
Solutions

1. Suppose you throw a ball vertically upward. If you release the ball 7 feet above the ground at an initial speed of 48 feet per second, how high will the ball travel? (Assume gravity is $-32 ft/sec^2$)

   We know that $a(t) = -32$, $v(0) = 48$, and $s(0) = 7$.

   Antidifferentiating, $v(t) = -32t + C$, so $v(0) = 48 = -32(0) + C$, so $C = 48$, and $v(t) = -32t + 48$.

   Antidifferentiating again, $s(t) = -16t^2 + 48t + D$, so $s(0) = 7 = D$, and $s(t) = -16t^2 + 48t + 7$.

   The max height ocurs when $v(t) = 0$, that is, when $-32t + 48 = 0$, or $32t = 48$, so when $t = \frac{48}{32} = \frac{3}{2}$. [Notice $s''(t) = a(t) < 0$, so we know it is a maximum]

   Thus, the max height is: $s(\frac{3}{2}) = -16 \left(\frac{3}{2}\right)^2 + 48 \left(\frac{3}{2}\right) + 7 = 43$, or 43 feet.

2. Use Newton's Method to approximate a real root of the function $f(x) = x^3 - 5x^2 + 27$ to 5 decimal places.

   Notice that $f(0) = 27$, and $f(-2) = -8 - 20 + 27 = 1$, so there is a root between -2 and 0.

   Also, $f'(x) = 3x^2 - 10x$

   Using Newton's method, recall that $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, so if we take $x_0 = -1$:

   $x_1 = -1 - \frac{-1-5+27}{3+10} \approx -2.615384615$

   $x_2 \approx -2.615384615 - \frac{f(x_1)}{f'(x_1)} \approx -2.0778105$

   $x_3 \approx -2.0778105 - \frac{f(x_2)}{f'(x_2)} \approx -1.9723552$

   $x_4 \approx -1.9723552 - \frac{f(x_3)}{f'(x_3)} \approx -1.9684133$

   $x_5 \approx -1.9723552 - \frac{f(x_4)}{f'(x_4)} \approx -1.9684079$

   $x_6 \approx -1.9684979 - \frac{f(x_5)}{f'(x_5)} \approx -1.9684079$

   Therefore, the root is at $x \approx -1.96841$

3. Use Newton's Method to approximate $\sqrt{10}$ to 5 decimal places.

   Notice that $\sqrt{10}$ is a root of the polynomial function $f(x) = x^2 - 10$.

   Also, $f'(x) = 2x$

   Using Newton's method, recall that $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, so if we take $x_0 = 3$:

   $x_1 = 3 - \frac{-1}{6} \approx 3.16666667$

   $x_2 \approx 3.16666667 - \frac{f(x_1)}{f'(x_1)} \approx 3.16228070$

   $x_3 \approx 3.16228070 - \frac{f(x_2)}{f'(x_2)} \approx 3.16227766$

   $x_4 \approx 3.16227766 - \frac{f(x_3)}{f'(x_3)} \approx 3.16227766$

   Therefore, the root is at $x \approx 3.16227766$

4. Find each of the following indefinite integrals:

(a) $\int \dfrac{x^{\frac{3}{2}} - 7x^{\frac{1}{2}} + 3}{x^{\frac{1}{2}}} dx$

$= \int x - 7 + 3x^{-\frac{1}{2}} dx = \dfrac{1}{2}x^2 - 7x + 6x^{\frac{1}{2}} + C$

(b) $\int \sin^3 x \cos x \, dx$

Let $u = \sin x$. Then $du = \cos x \, dx$, and we have $\int u^3 \, du = \dfrac{1}{4}u^4 + C$.

Thus the indefinite integral is: $\frac{1}{4}\sin^4 x + C$.

(c) $\int 5x(x^2 + 1)^8 dx$

Let $u = x^2 + 1$. Then $du = 2x \, dx$, or $\frac{1}{2} du = dx$, so we have $\int \dfrac{5}{2}u^8 du = \dfrac{5}{2}\dfrac{1}{9}u^9 + C = \dfrac{5}{18}u^9$.

Thus the indefinite integral is: $\frac{5}{18}(x^2 + 1)^9 + C$.

(d) $\int \dfrac{x}{\sqrt{x+1}} dx$

This is a trickier substituion problem. Let $u = x + 1$. Then $u - 1 = x$, and $du = dx$.

Thus we have the indefinite integral: $\int \dfrac{u-1}{\sqrt{u}} du = \int u^{\frac{1}{2}} - u^{-\frac{1}{2}} du = \dfrac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C$.

Thus the indefinite integral is: $\frac{2}{3}(x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} + C$.

5. Solve the following differential equations under the given initial conditions:

(a) $\frac{dy}{dx} = \sin x + x^2; y = 5$ when $x = 0$

Antidifferentiating, $y = -\cos x + \frac{1}{3}x^3 + C$.

Therefore, $5 = -\cos(0) + \frac{1}{3}(0)^3 + C$, or $5 = -1 + 0 + C$, so $C = 6$.

Therefore, $y = -\cos x + \frac{1}{3}x^3 + 6$.

(b) $g''(x) = 4\sin(2x) - \cos(x); g'(\frac{\pi}{2}) = 3; g(\frac{\pi}{2}) = 6$

Antidifferentiating, $g'(x) = -2\cos(2x) - \sin x + C$

Therefore, $3 = -2\cos(\pi) - \sin\frac{\pi}{2} + C = -2(-1) - 1 + C$, so $3 = 2 - 1 + C$, or $C = 2$.

Hence $g'(x) = -2\cos(2x) - \sin x + 2$. But then $g(x) = -\sin(2x) + \cos x + 2x + D$.

Moreover, $6 = -\sin(\pi) + \cos\frac{\pi}{2} + 2\left(\frac{\pi}{2}\right) + D$, or $6 = 0 + 0 + \pi + D$, so $D = 6 - \pi$

Thus $g(x) = -\sin(2x) + \cos x + 2x + 6 - \pi$.

6. Express the following in summation notation:

(a) $2 + 5 + 10 + 17 + 26 + 37 = \displaystyle\sum_{k=1}^{6} k^2 + 1$

(b) $x^2 + \frac{x^3}{4} + \frac{x^4}{9} + ... + \frac{x^{11}}{100} = \displaystyle\sum_{k=1}^{10} \dfrac{x^{k+1}}{k^2}$

7. Evaluate the following sums:

(a) $\displaystyle\sum_{k=2}^{5} k^2(k+1)$

$\displaystyle = \sum_{k=2}^{5} k^3 + k^2 = (8+4) + (27+9) + (64+16) + (125+25) = 278$

(b) $\displaystyle\sum_{k=3}^{20} k^3 - k^2$

$\displaystyle = \sum_{k=1}^{20} k^3 - \sum_{k=1}^{20} k^2 - \sum_{k=1}^{2} k^3 + \sum_{k=1}^{2} k^2$

$\displaystyle = \left(\frac{(20)(21)}{2}\right)^2 - \frac{(20)(21)(41)}{6} - 1 - 8 + 1 + 4 = (210)^2 - 2870 - 9 + 5 = 41,226$

8. Express the following sums in terms of n:

(a) $\displaystyle\sum_{k=1}^{n} 3k^2 - 2k + 10$

$\displaystyle = 3\sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 10 = 3\frac{(n)(n+1)(2n+1)}{6} - 2\frac{(n)(n+1)}{2} + 10n$

$\displaystyle = n^3 + \frac{3n^2}{2} + \frac{n}{2} - n^2 - n + 10n = n^3 + \frac{n^2}{2} + \frac{19n}{2}$

(b) $\displaystyle\sum_{k=3}^{n} k^2(k-1)$

$\displaystyle = \sum_{k=1}^{n} k^3 - \sum_{k=1}^{n} k^2 - \sum_{k=1}^{2} k^3 + \sum_{k=1}^{2} k^2$

$\displaystyle = \left(\frac{(n)(n+1)}{2}\right)^2 - \frac{(n)(n+1)(2n+1)}{6} - 1 - 8 + 1 + 4 = \frac{n^4 + 2n^3 + n^2}{4} - \frac{2n^3 + 3n^2 + n}{6} - 4 = \frac{n^4}{4} + \frac{n^3}{6} - \frac{n^2}{4} - \frac{n}{6} - 4.$

9. Consider $f(x) = 3x^2 - 5$ in the inteval $[3, 7]$

(a) Find a summation formula that gives an estimate the definite integral of f on $[3, 7]$ using n equal width rectangles and using midpoints to give the height of each rectangle. You do not have to evaluate the sum or find the exact area.

Notice that $\Delta x = \frac{7-3}{n} = \frac{4}{n}$. Since we want to use midpoints for our heights, $x_k = 3 + k\Delta x - \frac{\Delta x}{2} = 3 + \frac{4k-2}{n}$.

Therefore, $\displaystyle A_n = \sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n} \left[3\left(3 + \frac{4k-2}{n}\right)^2 - 5\right]\left(\frac{4}{n}\right)$

(b) Find the norm of the partition $P : 3 < 3.5 < 5 < 6 < 6.25 < 7$

The norm is the widest gap in the partition: $5 - 3.5 = 1.5$

(c) Find the approximation of the definite integral of f on $[3, 7]$ using the Riemann sum for the partition P given in part (b).

$A \approx \sum_{k=1}^{5} f(x_k)\Delta x_k = f(3.25)(.5) + f(4.25)(1.5) + f(5.5)(1) + f(6.125)(.25) + f(6.625)(.75) = (26.6875)(.5) + (49.1875)(1.5) + (85.75)(1) + (107.546875)(.25) + (126.671875)(.75) = 294.765625$

10. Assume f is continuous on $[-5, 3]$, $\int_{-5}^{-1} f(x)\, dx = -7$, $\int_{-1}^{3} f(x)\, dx = 4$, and $\int_{1}^{3} f(x)\, dx = 2$. Find:

(a) $\displaystyle\int_{3}^{-1} f(x)\, dx = -\int_{-1}^{3} f(x)\, dx = -4$

(b) $\displaystyle\int_{-5}^{1} f(x)\, dx = \int_{-5}^{-1} f(x)\, dx + \int_{-1}^{3} f(x)\, dx + \int_{3}^{1} f(x)\, dx = -7 + 4 - 2 = -5$

(c) $\displaystyle\int_{-5}^{3} f(x)\, dx = \int_{-5}^{-1} f(x)\, dx + \int_{-1}^{3} f(x)\, dx = -7 + 4 = -3$

(d) $\displaystyle\int_{-1}^{-1} f(x)\, dx = 0$

(e) Find the average value of f on $[-5, -1]$

$$= \frac{1}{-1 - (-5)}\int_{-5}^{-1} f(x)\, dx = \frac{1}{4}\cdot(-7) = -\frac{7}{4}$$

11. Evaluate the following:

(a) $\displaystyle\int_{1}^{4} x^3 + \frac{1}{\sqrt{x}} + 2\ dx$

$= \frac{1}{4}x^4 + 2x^{\frac{1}{2}} + 2x\big|_1^4 = \left[\frac{1}{4}4^4 + 2\cdot 4^{\frac{1}{2}} + 2(4)\right] - \left[\frac{1}{4}1^4 + 2\cdot 1^{\frac{1}{2}} + 2(1)\right] = 76 - 4.25 = 71.75$

(b) $\displaystyle\int_{0}^{1} x^2(2x^3 + 1)^2\ dx$

Let $u = 2x^3 + 1$. Then $du = 6x^2$, or $\frac{1}{6}du = dx$.

Notice $2(0)^3 + 1 = 1$, and $2(1)^3 + 1 = 3$

Then we have $\displaystyle\int_{1}^{3} \frac{1}{6}u^2\ du = \frac{1}{18}u^3\big|_1^3 = \frac{1}{18}(3^3 - 1^3) = \frac{26}{18} = \frac{13}{9}$.

(c) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^3(2x)\cos(2x)\ dx$

Let $u = \sin(2x)$. Then $du = 2\cos(2x)dx$ or $\frac{1}{2}du = \cos(2x)dx$

Notice that $\sin(2\frac{\pi}{6}) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, while $\sin\pi = 0$

Then we have $\displaystyle\int_{\frac{\sqrt{3}}{2}}^{0} \frac{1}{2}u^3 du = \frac{1}{4}u^4\big|_{\frac{\sqrt{3}}{2}}^{0} = 0 - \frac{1}{8}\cdot\left(\frac{\sqrt{3}}{2}\right)^4 = -\frac{9}{128}$

(d) $\displaystyle\int_{-\pi}^{\pi} \sin x\ dx = 0$, since $\sin x$ is an odd function.

(e) $\displaystyle\frac{d}{dx}\left(\int_{1}^{3} t\sqrt{t^2 - 1}\right)\ dt = 0$, since a definite integral gives a constant, and the derivative of a constant is zero.

(f) $\displaystyle\int_{1}^{3}\left[\frac{d}{dx}\left(t\sqrt{t^2 - 1}\right)\ dt\right] = (3\sqrt{3^2 - 1}) - (1\sqrt{1^2 - 1}) = 3\sqrt{8}$

12. Suppose $G(x) = \int_2^x \frac{1}{t^2+1}\, dt$

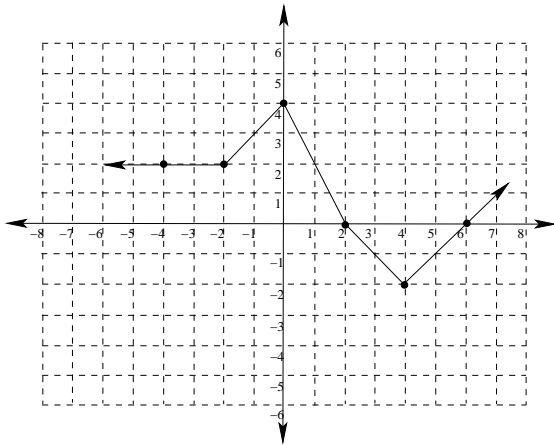   (a) Find $G'(2) = \frac{1}{2^2+1} = \frac{1}{5}$

    (b) Find $G'(x^2) = \frac{1}{x^4+1}$ [Note: $G'(x^2) \neq \frac{d}{dx}G(x^2)$]

    (c) Find $G''(3)$

        Notice that $\frac{d}{dx}\frac{1}{x^2+1} = \frac{-2x}{(x^2+1)^2}$.

        Thus $G''(3) = \frac{-2(3)}{(3^2+1)^2} = \frac{-6}{(10)^2} = \frac{-3}{50}$.

13. Given the following graph of $f(x)$ and the fact that $G(x) = \int_{-2}^t f(t)\, dt$:

    (a) Find $G(6) = 2.5 + 3.5 + \frac{1}{2}(2)(4) - \frac{1}{2}(4)(2) = 6$ [Compute area directly]

    (b) Find $G'(6) = f(6) = 0$

    (c) Find $G''(6) = f'(6) = 1$ [Compute slope of line segment on graph at $x = 6$]

14. (a) Use the Trapeziodal Rule with $n = 4$ to approximate $\int_0^4 2x^3\, dx$

    $A \approx \frac{4-0}{4\cdot 2}[f(0)+2f(1)+2f(2)+2f(3)+f(4)] = \frac{1}{2}[0+2(2\cdot1)+2(2\cdot8)+2(2\cdot27)+2\cdot64] = \frac{1}{2}[272] = 136$

    (b) Find the maximum possible error in your approximation from part (a).

        Notice that $f'(x) = 6x^2$ and $f''(x) = 12x$. This has a maximum of 48 when $x = 4$ on the interval $[0,4]$.

        Therefore, Error $\leq \frac{M(b-a)^3}{12n^2} = \frac{48(4)^3}{12\cdot4^2} = 16$.

    (c) Use the Fundamental Theorem of Calculus to find $\int_0^4 2x^3\, dx$ exactly. How far off was your estimate? How does the actual error compare to the maximum possible error?

        $\int_0^4 2x^3\, dx = \frac{1}{2}x^4\big|_0^4 = \frac{1}{2}4^4 - 0 = 128$

        The actual error is 136 - 128 = 8, which is half of the maxumum possible error for $n = 4$.