Math 261
Exam 4
11/30/2007

Name:____

Instructions: You will have 50 minutes to complete this exam. Calculators are allowed, but this is a closed book, closed notes exam. The credit given on each problem will be proportional to the amount of correct work shown. Correct answers without supporting work will receive little credit. Simplify answers when possible and follow directions carefully on each problem.

1. (5 points each) Test track trials have shown that a DeLorian can accelerate from 0 to 60 miles per hour in 6 seconds.

   (a) Assuming that the DeLorian undergoes constant acceleration, find a function $v(t)$ that gives the velocity of the DeLorian in $\frac{ft}{s}$ as a function of time, in seconds.
   Recall that if acceleration is constant, then $a(t) = k$, so, antidifferentiating, $v(t) = kt + C$
   Also, since the car begins at rest, $v(0) = 0$, so $C = 0$. If we convert $60mph$ to feet per second, we get:
   $\frac{60miles}{1hour} \cdot \frac{5280ft}{1mile} \cdot \frac{1hour}{60min} \cdot \frac{1min}{60sec} = 88\frac{ft}{sec}$.
   Then $v(6) = 88 = 6k$, so $k = \frac{88}{6} = \frac{44}{3}$, and $v(t) = \frac{44}{3}t\frac{ft}{s}$.

   (b) If Marty gets into a DeLorian parked at the Twin Pines Mall and accelerates toward 90 mph, how long will it take him to get the DeLorian up to exactly 88 mph?
   Again converting to feet per second: $\frac{88miles}{1hour} \cdot \frac{5280ft}{1mile} \cdot \frac{1hour}{60min} \cdot \frac{1min}{60sec} = \frac{1936}{15}\frac{ft}{sec}$.
   Solving for t, we see $\frac{44}{3}t = \frac{1936}{15}$ when $660t = 5808$, or $t = 8.8$ seconds.

   (c) If there is a photomat exactly 400 feet in front of his starting location and he drives straight toward it, will he get to 88 mph before running into the photomat? Justify your answer.
   Antidifferentiating $v(t) = \frac{44}{3}t\frac{ft}{s}$, we get $s(t) = \frac{22}{3}t^2\frac{ft}{s}$.
   Notice that $s(8.8) = \frac{22}{3}(8.8)^2 = 567.9$ feet. Since this is greater than 400 feet, Marty will not be able to get to 88 mph before running into the photomat.

2. (4 points each) Compute the following:

   (a) $\frac{d}{dx}\left(\int_1^{x^2} \frac{t^2}{t^2+1}\,dt\right)$

   $= \frac{d}{dt}\left(F(x^2) - F(1)\right) = F'(x^2) \cdot (2x)$

   $= \frac{(x^2)^2}{(x^2)^2+1} \cdot (2x) = \frac{2x^5}{x^4+1}$

   (b) $\int \frac{d}{dt}\left(\frac{t^2}{t^2+1}\right)\,dt$

   $= \frac{t^2}{t^2+1} + C$

   (c) $\frac{d}{dx}\left(\int_1^2 \frac{t^2}{t^2+1}\,dt\right)$

   Since $\int_1^2 \frac{t^2}{t^2+1}\,dt$ is a constant

   $\frac{d}{dx}\left(\int_1^2 \frac{t^2}{t^2+1}\,dt\right) = 0$

   (d) $\int_1^2 \left[\frac{d}{dx}\left(\frac{t^2}{t^2+1}\right)\,dt\right]$

   $= \frac{2^2}{2^2+1} - \frac{1^2}{1^2+1} = \frac{4}{5} - \frac{1}{2} = \frac{8}{10} - \frac{5}{10} = \frac{3}{10}$

3. (5 points each)

(a) Evaluate the sum $\sum_{k=3}^{10} k^2 + 5$

$$= \sum_{k=1}^{10} k^2 - \sum_{k=1}^{2} k^2 + \sum_{k=3}^{10} 5 = \frac{(10)(11)(21)}{6} - 1 - 4 + 8(5) = 385 - 5 + 40 = 420$$

(b) Express the following sum in terms of n: $\sum_{k=1}^{n} k^2 - 2k$

$$= \sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} k = \frac{(n)(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} = \frac{2n^3 + 3n^3 + n}{6} - n^2 - n = \frac{n^3}{3} - \frac{n^2}{2} - \frac{5n}{6}$$

4. (4 points each) Given $\int_1^5 g(x)\,dx = 6$, $\int_5^{11} g(x)\,dx = 3$, and $\int_5^7 g(x)\,dx = 2$, find:

(a) $\int_1^{11} g(x)\,dx$

$$= \int_1^5 g(x)\,dx + \int_5^{11} g(x)\,dx$$

$$= 6 + 3 = 9$$

(b) $\int_7^{11} g(x)\,dx$

$$= \int_5^{11} g(x)\,dx - \int_5^7 g(x)\,dx$$

$$= 3 - 2 = 1$$

(c) $\int_5^1 g(x)\,dx$

$$= -\int_1^5 g(x)\,dx = -6$$

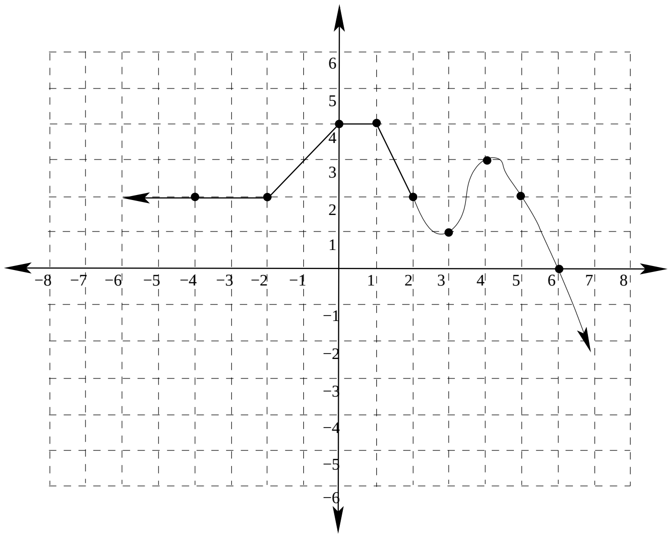
(d) $\int_7^7 g(x)\,dx$

$$= 0$$

(e) Assuming g is continuous, find the average value of g on the interval $[5, 11]$.

$$= \frac{1}{11-5}\int_5^{11} g(x)\,dx = \frac{1}{6} \cdot 3 = \frac{1}{2}$$

5. Given the following graph of $f(x)$



(a) (4 points) Find $\int_{-2}^{2} f(x)\, dx$ exactly

Using geometry, $\int_{-2}^{2} f(x)\, dx = 2.5 + 3.5 + 4 + 3 = 13$

(b) (6 points) Approximate $\int_{0}^{6} f(x)\, dx$ using 6 equal width rectangles with height given by the right hand endpoints.

First, $\Delta x = \frac{6-0}{6} = 1$.
Using right hand endpoints in 6 rectangles each of width 1:
$A \approx \Delta x \,[f(1) + f(2) + f(3) + f(4) + f(5) + f(6)]$
$= 1[4 + 2 + 1 + 3 + 2 + 0] = 12$

6. (8 points each) Evaluate each of the following:

(a) $\displaystyle\int \frac{3x^2 - 2}{(2x^3 - 4x)^5}\, dx$

Let $u = 2x^3 - 4x$. Then $du = 6x^2 - 4$ so $\frac{1}{2}\, du = 3x^2 - 2\, dx$

This gives $\displaystyle\frac{1}{2}\int \frac{1}{u^5}\, dx = \frac{1}{2} \cdot \frac{1}{-4} u^{-4} + C = -\frac{1}{8u^4} = -\frac{1}{8(2x^3 - 4x)^4} + C$

(b) $\displaystyle\int \frac{x}{\sqrt{x-2}}\, dx$

Let $u = x - 2$. Then $x = u + 2$, and $du = dx$.

Therefore, we have $\displaystyle\int \frac{u+2}{\sqrt{u}}\, du = \int u^{\frac{1}{2}} 2u^{-\frac{1}{2}}\, du = \frac{2}{3} u^{\frac{3}{2}} + 4u^{\frac{1}{2}} + C$

$\displaystyle= \frac{2}{3}(x-2)^{\frac{3}{2}} + 4(x-2)^{\frac{1}{2}} + C$

(c) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin x \cos^2(x)\, dx$

Let $u = \cos x$. Then $du = -\sin x\, dx$, or $-du = \sin x\, dx$

Notice $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$, and $\cos(\frac{\pi}{2}) = 0$

Thus, we have $\displaystyle-\int_{\frac{\sqrt{3}}{2}}^{0} u^2\, du = -\frac{1}{3}u^3 \Big|_{\frac{\sqrt{3}}{2}}^{0} = 0 - \left[-\frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)^3\right] = \frac{1}{3} \cdot \frac{3\sqrt{3}}{8} = \frac{\sqrt{3}}{8}$

7. (6 points each)

(a) Use the Trapezoidal Rule with $n = 4$ to approximate $\displaystyle\int_{-2}^{2} \left(x^4 - 2x^2\right)\, dx$

First, $\Delta x = \frac{2-(-2)}{4} = 1$.
Then $A \approx \frac{\Delta x}{2}\left[f(-2) + 2f(-1) + 2f(0) + 2f(1) + f(2)\right]$
$= \frac{1}{2}\left[((-2)^4 - 2(-2)^2) + 2((-1)^4 - 2(-1)^2) + 2((0)^4 - 2(0)^2) + 2((1)^4 - 2(1)^2) + ((2)^4 - 2(2)^2)\right]$
$= \ [8 - 2 + 0 - 2 + 8] = \frac{1}{2} \cdot 12 = 6$

(b) Estimate the maximum error in your approximation from part (a).

Recall, $Error \leq \frac{M(b-a)^3}{12n^2}$, where M is a value such that $|f''(x)| \leq M$ for all $x \in [a, b]$. Here, $a = -2$, $b = 2$, and $n = 4$.
Now, $f'(x) = 4x^3 - 4x$, $f''(x) = 12x^2 - 4$, and $f'''(x) = 12x$, which has a single critical point at $x = 0$.
Therefore, to find the maximum and minimum values of $f''(x)$ we need to look at:
$f''(-2) = 44$
$f''(0) = -4$
and $f''(2) = 44$
Thus $M = 44$, and $Error \leq \frac{44(2-(-2))^3}{12(4)^2} = \frac{44}{3}$.

Extra Credit: (5 points) Determine the minimum number of rectangles should be used in order to guarantee an approximation of $\displaystyle\int_{-2}^{2} \left(x^4 - 2x^2\right)\, dx$ is accurate to within .001 when using the Trapezoid Rule.

Here, we need $\frac{M(b-a)^3}{12n^2} \leq .001$, or $\frac{44(4)^3}{12n^2} \leq .001$

Then $44(4)^3 \leq .001(12n^2)$, or $\frac{44(4)^3}{(.001)(12)} \leq n^2$.

Therefore, $234666\frac{2}{3} \leq n^2$, so $n \approx 484.424$

Thus to guarantee an estimate within .001 of the actual value, we need to take n at least 485.