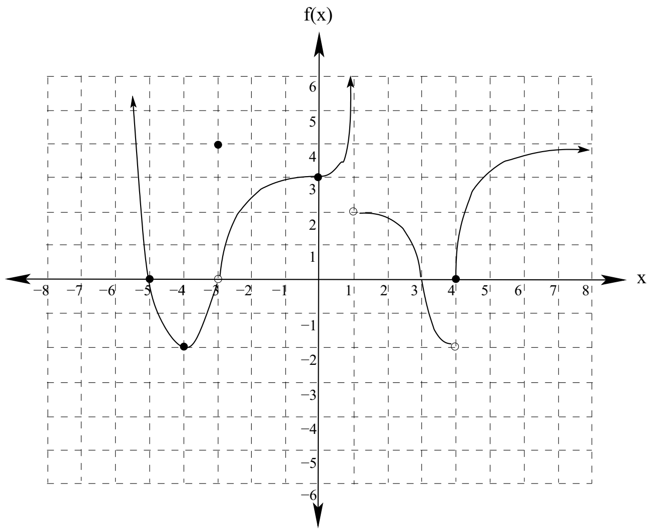
Math 261
Final Exam - Practice Problem Solutions

1. A function f is graphed below.



   (a) Find $f(0)$, $f(-2)$, $f(1)$, and $f(4)$
   $f(0) = 3$; $f(-2) \approx 2.4$; $f(1)$ is undefined; $f(4) = 0$

   (b) Find the domain and range of f
   Domain: $(-\infty, 1) \cup (1, \infty)$ Range: $[-2, \infty)$

   (c) Find the intervals where $f'(x)$ is positive
   $f'(x) > 0$ on $(-4, 1) \cup (4, \infty)$

   (d) Find the intervals where $f''(x)$ is negative.
   $f''(x) < 0$ on $(-3, 0) \cup (1, 3) \cup (4, \infty)$

   (e) Find $\lim_{x \to -2} f(x)$
   $\lim_{x \to -2} f(x) \approx 2.4$

   (f) find $\lim_{x \to 4^-} f(x)$ and $\lim_{x \to 4^+} f(x)$
   $\lim_{x \to 4^-} f(x) = -2$; $\lim_{x \to 4^+} f(x) = 0$

   (g) find $\lim_{x \to -1^-} f(x)$ and $\lim_{x \to -1^+} f(x)$
   $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) \approx 2.9$

   (h) find $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$
   $\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = 4$

   (i) find the points where $f(x)$ is discontinuous, and classify each point of discontinuity.
   f has a removable discontinuity at $x = -3$, an infinite discontinuity at $x = 1$, and a jump discontinuity at $x = 4$.

2. Evaluate the following limits:

   (a) $\lim_{x \to 1} \dfrac{2x^2 - 5x - 3}{3x^2 - 4x - 15} = \lim_{x \to 1} \dfrac{(2x+1)(x-3)}{(3x+5)(x-3)} = \dfrac{(2x+1)}{(3x+5)} = \dfrac{3}{8}$

   (b) $\lim_{x \to 3} \dfrac{2x^2 - 5x - 3}{3x^2 - 4x - 15} = \lim_{x \to 3} \dfrac{(2x+1)(x-3)}{(3x+5)(x-3)} = \dfrac{(2x+1)}{(3x+5)} = \dfrac{7}{10}$

   (c) $\lim_{x \to 2} \sqrt{2x - 4}$ is undefined since $\sqrt{2x - 4}$ is only defined for $x \geq 2$.

(d) $\lim_{x\to\pi} \cos x = -1$

(e) $\lim_{x\to\infty} \cos x$ is undefined ($\cos x$ continues to oscillate from 1 to -1 and back)

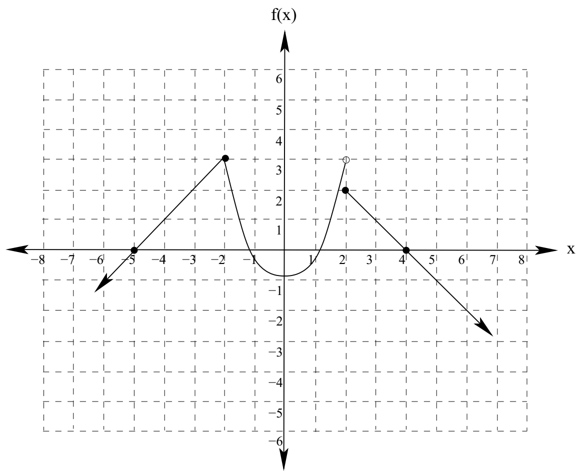
(f) $\lim_{x\to\infty} \dfrac{2x^2-5x-3}{3x^2-4x-15} = \dfrac{2}{3}$

(g) $\lim_{x\to\infty} \dfrac{2x^2-5x-3}{3x^3-4x-15} = 0$

3. Given the function

$$f(x) = \begin{cases} x+5 & \text{if } x \le -2 \\ x^2-1 & \text{if } |x| < 2 \\ 4-x & \text{if } x \ge 2 \end{cases}$$

(a) Graph $f(x)$.



(b) Find $\lim_{x\to 2^-} f(x)$, $\lim_{x\to 2^+} f(x)$, and $\lim_{x\to -2} f(x)$

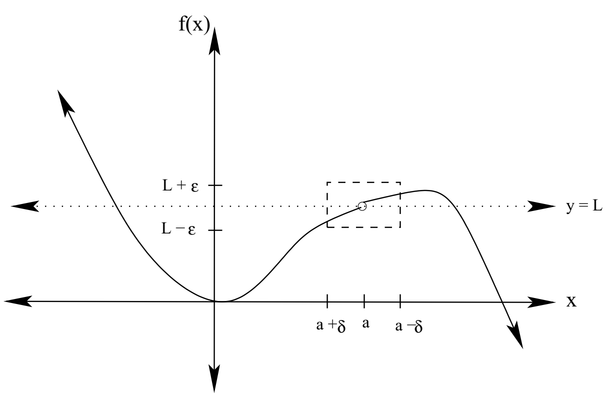
$\lim_{x\to 2^-} f(x) = 3$, $\lim_{x\to 2^+} f(x) = 2$, and $\lim_{x\to -2} f(x) = 3$

(c) Is $f(x)$ continuous at $x = 1$? Justify your answer.

Yes. In an interval containing $x = 1$, the function f is defined by $x^2 - 1$ which is a polynomial and hence is continuous.

4. Give the formal ϵ - δ definition of the linit of a function as presented in class. Then draw a diagram illustrating the definition. Finally, write the definition informally in your own words.

Let f be defined on an open interval containing a, except possibly at a itself, and let L be a real number. The statement $\lim_{x\to a} f(x) = L$ means that for every $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$.



Intuitively, what the formal definition of a limit says is that $\lim_{x\to a} f(x) = L$ means that if we set an error tolerance of ϵ on the y-axis, then no matter how small we set our error tolerance, it is possible to choose an error tolerance δ on the x-axis so that all points within δ of a on the x-axis get mapped by the function to points that are within ϵ of our limit value L.

5. Given that $f(x) = 3x^2 - 1$, $\lim_{x\to 1} f(x) = 2$, and $\epsilon = .01$, find the largest δ such that if $0 < |x-1| < \delta$, then $|f(x) - 2| < \epsilon$.

   We need $|f(x) - 2| < \epsilon$. That is $|3x^2 - 1 - 2| < .01$ or $|3x^2 - 3| < .01$

   Therefore, $-.01 < 3x^2 - 3 < .01$, or $2.99 < 3x^2 < 3.01$, so $\frac{2.99}{3} < x^2 < \frac{3.01}{3}$.

   Hence $\sqrt{\frac{2.99}{3}} < x < \sqrt{\frac{3.01}{3}}$ or, rounding, $.998331942 < x < 1.001166528$.

   Thus $-.00166806 < x - 1 < .00166528$. So we can take $\delta < \sqrt{\frac{3.01}{3}} - 1 \approx .00166528$

6. Use the formal definition of a limit to prove that $\lim_{x\to 2} 5 - 2x = 1$.

   Suppose $|f(x) - L| < \epsilon$. Then $|5 - 2x - 1| < \epsilon$ or $|4 - 2x| < \epsilon$.

   That is, $|2(2-x)| < \epsilon$, or $2|2-x| < \epsilon$, so $|x-2| < \frac{\epsilon}{2}$.

   Let $\delta \le \frac{\epsilon}{2}$. Then if $0 < |x-2| < \delta \le \frac{\epsilon}{2}$, $2|x-2| < \epsilon$.

   Therefore $|2x - 4| = |4 - 2x| = |5 - 2x - 1| < \epsilon$. That is, $|f(x) - 1| < \epsilon$.

   Hence $\lim_{x\to 2} f(x) = 1$

7. Let $f(x) = \dfrac{2x^2 - 4x}{x^2 - x - 2}$.

   (a) Find the values of x at which f is discontinuous.

   Notice that $f(x) = \dfrac{2x^2 - 4x}{x^2 - x - 2} = \dfrac{2x(x-2)}{(x-2)(x+1)}$.

   Since f is a rational function, it is continuous everywhere except where the denominator is zero. Therefore, f is discontinuous at $x = 2$ and at $x = -1$.

   (b) Find all vertical and horizonal asymptotes of f.

   Since the disontinuity at $x = 2$ is removable, f only has one vertical asymptote at $x = -1$.

   Since $\lim_{x\to\infty} f(x) = 2$, f has a horizontal asymptote at $y = 2$.

8. Find the x values at which $f(x) = \sqrt{3 - 2x} + \dfrac{1}{\sqrt{2x+5}}$ is continuous.

   Since f consists of a sum of sqare roots of polynomial functions, f is continuous wherever it is defined. Thus we need $3 - 2x \ge 0$, or $x \le \frac{3}{2}$ and we need $2x + 5 > 0$, or $x > -\frac{5}{2}$. Thus f is continuous on the interval $(-\frac{5}{2}, \frac{3}{2}]$.

9. (a) Use the Intermediate Value Theorem to show $f(x) = 2x^3 + 3x - 4$ has a root between 0 and 1.

   First notice that f is a polynomial, and hence is continuous everywhere, so the IVT applies in this situation. Next, $f(0) = -4$, and $f(1) = 1$, so by the IVT, f attains every value between -4 and 1 at leat once for some x-value in the interval $(0, 1)$. In particular, there must be some c in the interval $(0, 1)$ with $f(c) = 0$.

   (b) Use Newton's method to approximate this root to 4 decimal places.

   Recall that Newton's method uses the derivative to recursively approximate a root of a function. Given an initial guess x_0, we compute approximations using the formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

   Here, $f'(x) = 6x^2 + 3$, and we will take $x_0 = .5$.

   Then $x_1 = 1$; $x_2 = .8888889$, $x_3 = .8796739$, $x_4 = .87961488$, $x_5 = .87961488$, so our approximation of the root to four decimal places is :$x = .8796$.

10. Find the derivative $y' = \frac{dy}{dx}$ for each of the following:

   (a) $y = \pi^3 + \pi^2 x - \pi x^3 + x^\pi$
   $y' = \pi^2 - 3\pi x^2 + \pi x^{\pi - 1}$

   (b) $y = \cos(3x) + \sin(3x)$
   $y' = -3\sin(3x) + 3\cos(3x)$

   (c) $y = x^4 + \cos(x^4)$
   $y' = 4x^3 - 4x^3 \sin(x^4)$

(d) $y = \sqrt{x}\tan x$

$y' = \frac{1}{2}x^{-\frac{1}{2}}\tan x + x^{\frac{1}{2}}\sec^2(x)$

(e) $y = \sec^3(x^3)$

$y' = 3\sec^2(x^3)\cdot\sec(x^3)\tan(x^3)\cdot 3x^2 = 9x^2\sec^3(x^3)\tan(x^3)$

(f) $y = \frac{3-x}{x^2+1}$

$y' = \frac{(1)(x^2+1)-(3-x)(2x)}{(x^2+1)^2} = \frac{x^2-6x-1}{(x^2+1)^2}$.

(g) $y = \dfrac{x^2\cos x}{x+\sin(3-2x)}$

$y' = \frac{(2x\cos x - x^2\sin x)(x+\sin(3-2x))-(x^2\cos x)(1-2\cos(3-2x))}{(x+\sin(3-2x))^2}$

(h) $y = \sin^2(\tan(x^3-5))$

$y' = 2\sin(\tan(x^3-5))\cos(\tan(x^3-5))\sec^2(x^3-5)\cdot 3x^2$

(i) $x^2 - 3xy + y^2 = 0$

Differentiating implicitly, $2x - 3y - 3xy' + 2yy' = 0$, so $2yy' - 3xy' = 3y - 2x$, or $y'(2y-3x) = 3y-2x$. Hence $y' = \frac{3y-2x}{2y-3x}$

(j) $2x^2y - 5xy - 3y^2 = 10$ Differentiating implicitly, $4xy + 2x^2y' - 5y - 5xy' - 6yy' = 0$, so $4xy - 5y = -2x^2y' + 5xy' + 6yy' = y'(-2x^2+5x+6y)$. Hence $y' = \frac{4xy-5y}{-2x^2+5x+6y}$

11. Use the formal limit definition of the derivative to find the derivative of the following:

(a) $f(x) = 3x^2 - x + 5$

$$f'(x) = \lim_{h\to 0}\frac{3(x+h)^2-(x+h)+5-(3x^2-x+5)}{h} = \lim_{h\to 0}\frac{3x^2+6xh+3h^2-x-h-3x^2+x-5}{h}$$

$$= \lim_{h\to 0}\frac{6xh+3h^2-h}{h} = \lim_{h\to 0} 6x+3h-1 = 6x-1$$

(b) $f(x) = \dfrac{2}{x-1}$

$$f'(x) = \lim_{h\to 0}\frac{\frac{2}{x+h-1}-\frac{2}{x-1}}{h} = \lim_{h\to 0}\frac{\frac{2(x-1)-2(x+h-1)}{(x+h-1)(x-1)}}{h}$$

$$= \lim_{h\to 0}\frac{2x-2-2x-2h+2}{(x+h-1)(x-1)}\frac{1}{h}$$

$$= \lim_{h\to 0}\frac{-2h}{(x+h-1)(x-1)}\frac{1}{h} = \lim_{h\to 0}\frac{-2}{(x+h-1)(x-1)} = \frac{-2}{(x-1)^2}$$

(c) $f(x) = \sqrt{x+1}$

$$f'(x) = \lim_{h\to 0}\frac{\sqrt{x+h+1}-\sqrt{x+1}}{h}\cdot\frac{\sqrt{x+h+1}+\sqrt{x+1}}{\sqrt{x+h+1}+\sqrt{x+1}}$$

$$= \lim_{h\to 0}\frac{x+h+1-(x+1)}{h(\sqrt{x+h+1}+\sqrt{x+1})} = \lim_{h\to 0}\frac{h}{h(\sqrt{x+h+1}+\sqrt{x+1})}$$

$$= \lim_{h\to 0}\frac{1}{\sqrt{x+h+1}+\sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}$$

12. Use the quotient rule to derive the formula for the derivative of $\sec(x)$.

Recall that $\sec x = \frac{1}{\cos x}$, so by the quotient rule:

$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \frac{0-(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \sec x\tan x$.