27. A particle is moving along a straight line during a sixty-second time interval. Every ten seconds the position, velocity, and acceleration of the particle are measured, and recorded in the table below. For each of the following questions, answer the question and justify your answer.

t (seconds)	0	10	20	30	40	50	60
$s(t)$ (feet)	0	15	20	35	80	50	20
$v(t)$ (ft/s)	0	2	3	4	2	3	0
$a(t)$ (ft/s^2)	2	1	0	1	0	5	1

(a) Is there a time when $s(t) = 45$ feet?

Yes. We may apply the IVT to the function $s(t)$ over the interval $[30, 40]$. Since $s(30) = 35 < 45 < s(40) = 80$ we may conclude that there is a c in the interval $(30, 40)$ with $s(c) = 45$ feet.

(b) Is there a time when $v(t) = 1$ ft/sec?

Yes. We may apply the IVT to the function $v(t)$ over the interval $[0, 10]$. Since $v(0) = 0 < 1 < v(10) = 2$ we may conclude that there is a c in the interval $(0, 10)$ with $v(c) = 1$ ft/sec.

(c) Is there a time when $v(t) = .5$ ft/sec?

Yes. We may apply the IVT to the function $v(t)$ over the interval $[0, 10]$. Since $v(0) = 0 < .5 < v(10) = 2$ we may conclude that there is a c in the interval $(0, 10)$ with $v(c) = .5$ ft/sec.

(d) Is there a time when $v(t) = -3$ ft/sec?

Since $v(t) \geq 0$ for all values in the table, the IVT will not help us here. However, we may apply the MVT to the function $s(t)$ over the interval $[40, 50]$. Since $\dfrac{s(50) - s(40)}{50 - 40} = \dfrac{80 - 50}{10} = -3$ we may conclude that there is a c in the interval $(40, 50)$ with $s'(c) = v(c) = -3$ ft/sec.

(e) What is the largest velocity that you can justify from the data in the table?

The largest value we can conclude using the IVT is 80 ft/sec. However, we may apply the MVT to the function $s(t)$ over the interval $[30, 40]$. Since $\dfrac{s(40) - s(30)}{40 - 30} = \dfrac{80 - 35}{10} = 4.5$ we may conclude that there is a c in the interval $(30, 40)$ with $s'(c) = v(c) = 4.5$ ft/sec.

(f) Is there a time when $a(t) = 3$ ft/sec^2?

Yes. We may apply the IVT to the function $a(t)$ over the interval $[40, 50]$. Since $a(40) = 0 < 3 < a(50) = 5$ we may conclude that there is a c in the interval $(0, 10)$ with $v(c) = .5$ ft/sec^2.

(g) Is there a time when $a(t) = -0.2$ ft/sec^2?

Since $a(t) \geq 0$ for all values in the table, the IVT will not help us here. However, we may apply the MVT to the function $v(t)$ over the interval $[30, 40]$. Since $\dfrac{v(40) - v(30)}{40 - 30} = \dfrac{2 - 4}{10} = -.2$ we may conclude that there is a c in the interval $(30, 40)$ with $v'(c) = a(c) = -.2$ ft/sec^2.

28. Find a number c in the given interval that satisfies the Mean Value Theorem for the function and interval given, or explain why the Mean Value Theorem does not apply.

(a) $f(x) = x^4 + 2x$ on $[-1, 1]$

First, notice that f is continuous on $[-1, 1]$ so the MVT applies. Also, $\dfrac{f(1) - f(-1)}{1 - (-1)} = \dfrac{3 - (-1)}{2} = 2$. Next, $f'(x) = 4x^3 + 2$, so we need to find x so that $4x^3 + 2 = 2$. That is, $4x^3 = 0$, or $x^3 = 0$. Thus $x = 0$ is the value that satisfies the Mean Value Theorem on the interval $[-1, 1]$.

(b) $f(x) = \dfrac{x^2}{x - 1}$ on $[-2, 2]$

Here, notice that $f(x)$ has a vertical asymptote when $x = 1$, thus the MVT does not apply.

29. A hotel that charges \$80 per day for a room gives special rates to organizations that reserve between 30 and 60 rooms. If more than 30 rooms are reserved, the charge per room is decreased by \$1 times the number of rooms over 30. Under these terms, what number of rooms gives the maximum income?

Let x be the number of rooms reserved. We know $0 \leq x \leq 60$, but since there is no discount for below 30 rooms, we know that $x \geq 30$.

The price per room for between 30 and 60 rooms is given by: $p(x) = 80 - (x - 30) = 110 - x$. Therefore, the revenue from x rooms is given by $R(x) = x(110 - x) = 110x - x^2$.

Now $R'(x) = 110 - 2x = 0$ when $110 = 2x$, or when $x = 55$.

Thus, applying the EVT, since $R(30) = 2400$, $R(55) = 3025$ and $R(60) = 3000$, so the maximum revenue occurs when 55 rooms are reserved.

30. A page of a book is to have an area of 90 square inches with 1 inch margins on the bottom and sides of the page and a 1/2-inch margin at the top. Find the dimensions of the page that would allow the largest printed area.

    Let x be the width of the page and y the length (in inches). Then $xy = 90$, or $y = \frac{90}{x}$. Accounting for the margins, we can see that the area of part of the page used for printing is given by: $A = (x-2)(y-\frac{3}{2}) = (x-2)(\frac{90}{x} - \frac{3}{2}) = 93 - \frac{180}{x} - \frac{3}{2}x^2$.
    Therefore, $A'(x) = \frac{180}{x^2} - \frac{3}{2} = 0$ when $\frac{180}{x^2} = \frac{3}{2}$, or $x^2 = \frac{360}{3} = 120$, so $x = \sqrt{120}$. Notice that $A''(x) = -\frac{360}{x^3} < 0$ for $x > 0$, so this is a maximum.

    Therefore the page should have dimensions $x = \sqrt{120} \approx 10.95$ inches and $y = \frac{90}{\sqrt{120}} \approx 8.22$ inches.

31. Sketch the following functions using information about the domain, the intercepts, the intervals where the function is increasing or decreasing, any local maximum or minimum, the intervals where it is concave up or concave down, any inflection points, and any asymptotes.

    (a) $f(x) = 2x^3 - 6x^2 - 18x$

    (b) $f(x) = \dfrac{x}{2x-1}$

    (c) $f(x) = \sin x + \cos x$

    [I'll add these in later if I have time. Sorry!]

32. Suppose you are standing on the roof of a shed that is 20 feet tall and you throw a ball vertically upward. If you release the ball 4 feet above the top of the shed at an initial speed of 40 feet per second, how high will the ball travel and when will the ball hit the ground? (Assume gravity is $-32ft/sec^2$)

    From the description above, we see that $s(0) = 24$, $v(0) = 40$, and $a(t) = -32$.

    Then $v(t) = -32t + 40$, and $s(t) = -16t^2 + 40t + 24$.

    The ball will attain its maximum height when $v(t) = 0$, that is when $32t = 40$, or when $t = \frac{5}{4}$. Then the maximum height is $s(\frac{5}{4}) = -16(\frac{5}{4})^2 + 40(\frac{5}{4} + 40 = -25 + 50 + 24 = 49$ feet.

    The ball will hit the ground when $s(t) = 0$, that is, when $-16t^2 + 40t + 24 = -8(2t^2 - 5t - 3) = -8(2t+1)(t-3) = 0$, which has solutions $t = -\frac{1}{2}$ and $t = 3$. Clearly the negative solution does not make sense, so the ball must hit the ground after 3 seconds.

33. Use Newton's Method to approximate a real root of the function $f(x) = x^3 - 3x^2 + 2$ to 5 decimal places.

    A clever student will notice that $f(1) = 0$, but that is not really using Newton's method, so we will pretend that we didn't notice this and apply Newton's method.

    Recall that Newton's method uses the derivative to recursively approximate a root of a function. Given an initial guess x_0, we compute approximations using the formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

    Here, $f'(x) = 3x^2 - 6x$, notice that $f'(x) = 0$ when $x = 0$ and when $x - 2$ so we ned to avoid these values (why?). We will take $x_0 = 1.5$.

    Then $x_1 = .888888889$; $x_2 = 1.000925926$, $x_3 = .999999999$, $x_4 = 1$ (we are probably rounding too soon here, but that's all the precision my calculator has), $x_5 = 1$, ..., so our approximation of the root to five decimal places is $:x = 1.00000$.

34. Use Newton's Method to approximate $\sqrt[3]{6}$ to 5 decimal places.

    The first trick here is to come up with a polynomial function that has $\sqrt[3]{6}$ as a root (well, I suppose it wouldn't have to be a polynomial function, but they are pretty easy to come by). If we start with $x = \sqrt[3]{6}$, then, cubing each side, we have $x^3 = 6$, so we may take $f(x) = x^3 - 6$.

    Then $f'(x) = 3x^2$.

    Newton's method uses the derivative to recursively approximate a root of a function. Given an initial guess x_0, we compute approximations using the formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

    Since we know that $1 < \sqrt[3]{6} < 2$, then we will take $x_0 = 2$.

    Then $x_1 = 1.833333333$; $x_2 = 1.817263545$, $x_3 = 1.817120604$, $x_4 = 1.817120593$, $x_5 = 1.817120593$, so our approximation of the root to five decimal places is $:x = 1.81712$.

35. Find each of the following indefinite integrals:

(a) $\displaystyle\int x^2 + C\,dx = \frac{1}{3}x^3 + Cx + D$

(b) $\displaystyle\int x^2 + C\,dC = x^2C + \frac{1}{2}C^2 + D$

(c) $\displaystyle\int \frac{3x^{\frac{3}{2}} - 5x^{\frac{1}{2}} + 3}{x^{\frac{1}{2}}}\,dx = \int 3x - 5 + 3x^{-\frac{1}{2}} = \frac{3}{2}x^2 - 5x + 6x^{\frac{1}{2}} + C$

(d) $\displaystyle\int x^2\sqrt{x^3 - 5}\,dx$

Let $u = x^3 - 5$. Then $du = 3x^2dx$, so $\frac{1}{3}du = x^2dx$.

This gives the integral $\displaystyle\frac{1}{3}\int u^{\frac{1}{2}}\,du = \frac{1}{3}\cdot\frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{9}u^{\frac{3}{2}} + C = \frac{2}{9}(x^3 - 5)^{\frac{3}{2}} + C.$

(e) $\displaystyle\int 4\tan^3 x\sec^2 x\,dx$

Let $u = \tan x$. Then $du = \sec^2 x dx$. This gives the integral $\displaystyle 4\int u^3\,du = u^4 + C = \tan^4 + C.$

(f) $\displaystyle\int \frac{x}{\sqrt{2x - 1}}\,dx$

Let $u = 2x - 1$. Then $du = 2dx$, and $\frac{u+1}{2} = \frac{u}{2} + \frac{1}{2} = x$.

This gives the integral $\displaystyle\int \frac{\frac{u}{2} + \frac{1}{2}}{u^{\frac{1}{2}}}\,du = \int \frac{1}{2}u^{\frac{1}{2}} + \frac{1}{2}u^{-\frac{1}{2}}\,du = \frac{1}{3}u^{\frac{3}{2}} + u^{\frac{1}{2}} + C.$

36. Solve the following differential equations under the given initial conditions:

(a) $f'(x) = \cos x + 2x; f(0) = 5$

Antidifferentiating, we have $f(x) = \sin x + x^2 + C$, so $f(0) = 5 = \sin 0 + 0^2 + C = C$, so $C = 5$.
Thus $f(x) = \sin x + x^2 + 5$

(b) $g''(x) = 4\cos(2x) - 6\sin(3x); g'(\pi) = 4; g(0) = 6$

Antidifferentiating, we have $g'(x) = 2\sin(2x) + 2\cos(3x) + C$, so $g'(\pi)4 = 2\sin(\pi) + 2\cos(\pi) + C = 0 - 2 + C$.
Therefore, $C = 6$, and $g'(x) = 2\sin(2x) + 2\cos(3x) + 6$
Next, $g(x) = -\cos(2x) + \frac{2}{3}\sin(3x) + D$, so $g(0) = 6 = -\cos(0) + \frac{2}{3}\sin(0) + D = -1 + D$. Thus $D = 7$
Hence $g(x) = -\cos(2x) + \frac{2}{3}\sin(3x) + 7$

37. Express the following in summation notation:

(a) $\displaystyle 2 + 5 + 8 + 11 + 14 + 17 = \sum_{k=1}^{6} 3k - 1$

(b) $\displaystyle \frac{2}{5} + \frac{3}{7} + \frac{4}{9} + \frac{5}{11} + \frac{6}{13} + \frac{7}{15} + \frac{8}{17} = \sum_{k=1}^{7}\frac{k+1}{2k+3}$

38. Evaluate the following sums:

(a) $\displaystyle\sum_{k=2}^{5} k(2k - 1)$

$= (8 - 2) + (18 - 3) + (32 - 4) + (50 - 5) = 5 + 15 + 28 + 45 = 94.$

(b) $\displaystyle\sum_{k=4}^{15} k^3 - 2k^2$

Here, it is probably better to use summation formulas, although you could certainly just calculate and then add up the terms.

Recall that $\displaystyle\sum_{k=1}^{n} k^3 = \left(\frac{(n)(n+1)}{2}\right)^2$ and $\displaystyle\sum_{k=1}^{n} k^2 = \left(\frac{n(n+1)(2n+1)}{6}\right).$

Then $\sum_{k=4}^{15} k^3 - 2k^2 = \sum_{k=1}^{15} k^3 - 2k^2 - \sum_{k=1}^{3} k^3 - 2k^2 = \left[\left(\frac{(15)(16)}{2}\right)^2 - 2\left(\frac{15(16)(31)}{6}\right)\right] - \left[\left(\frac{3(4)}{2}\right)^2 - 2\left(\frac{3(4)(7)}{6}\right)\right]$
$= \left[(120)^2 - (2480)\right] - \left[6^2 - 28\right] = 11,912$

39. Express the following sums in terms of n:

(a) $\sum_{k=1}^{n} k^3 - 3k + 5$
$= \left(\frac{n(n+1)}{2}\right)^2 - 3\left(\frac{n(n+1)}{2}\right) + 5n = \frac{n^4+2n^3+n^2}{4} - \frac{-3n^2-3n}{2} + 5n = \frac{n^2}{4} + \frac{n^3}{2} - \frac{5n^2}{4} + \frac{7n}{2}$

(b) $\sum_{3}^{n} k(3-k^2) = \sum_{3}^{n} 3k - k^3 = \sum_{1}^{n} 3k - k^3 - (3-1) - (6-8)$
$= 3\left(\frac{n(n+1)}{2}\right) - \left(\frac{n(n+1)}{2}\right)^2 - 2 + 2 = \left(\frac{3n^2}{2} + \frac{3n}{2}\right) - \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}\right) = -\frac{n^4}{4} - \frac{n^3}{2} + \frac{5n^2}{4} + \frac{3n}{2}$