Math 261
Integration Handout
11/20/2007

Antiderivatives and Integration

A. Antiderivatives

A function F is an **antiderivative** of f if $F'(x) = f(x)$.
Example: $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$, since $F'(x) = 3x^2 = f(x)$.
Note: If $F(x)$ is an antiderivative of f, then so is $F(x) + C$ for any constant C.

B. Indefinite Integrals

The notation $\int f(x)\ dx = F(x) + C$ is used to denote the indefinite integral of the function $f(x)$ with respect to the variable x. Here, $F(x) + C$ is the general antiderivative of $f(x)$.

C. Integration Rules

1. Integrating a constant: $\int a\ dx = ax + C$ for any constant a.

   Example: $\int \frac{3}{2}\ dx = \frac{3}{2}x + C$

2. Integrating Power Functions: $\int x^r\ dx = \frac{1}{r+1}x^{r+1} + C$

   Example: $\int x^{\frac{3}{2}}\ dx = \frac{2}{5}x^{\frac{5}{2}} + C$

3. Constant Multiples: $\int af(x)\ dx = a\int f(x)\ dx$ for any constant a.

   Example: $\int 7x^{\frac{3}{2}}\ dx = 7\int x^{\frac{3}{2}}\ dx = (7)\frac{2}{5}x^{\frac{5}{2}} + C = \frac{14}{5}x^{\frac{5}{2}} + C$

4. Sums and Differences: $\int f(x) + g(x)\ dx = \int f(x)\ dx + \int g(x)\ dx$,

   and $\int f(x) - g(x)\ dx = \int f(x)\ dx - \int g(x)\ dx$ for any functions f and g.

   Example: $\int 3x^2 + x^{\frac{3}{2}}\ dx = \int 3x^2\ dx + \int x^{\frac{3}{2}}\ dx = x^3 + \frac{2}{5}x^{\frac{5}{2}} + C$

D. Definite Integrals

Approximating Area Using Inscribed and Circumscribed Rectangles:

The area of a region under a continuous function $f(x)$ on an interval $[a, b]$ can be approximated using rectangles. If the rectangles are of equal width, then $\Delta x = \frac{b-a}{n}$ is the width of each rectangle, and $f(u_k)$ gives the minimum height (the height of the inscribed rectangle) in the kth subinterval, while $f(v_k)$ gives the maximum height (the height of the circumscribed rectangle) in the ith subinterval.

Then $A_{IP} = \sum_{k=1}^{n} f(u_k)\Delta x$ and $A_{CP} = \sum_{k=1}^{n} f(v_k)\Delta x$ give the estimate of the area by inscribed rectangles and circumscribed rectangles, respectively.

Morover, $A_{IP} = \sum_{k=1}^{n} f(u_k)\Delta x \leq A \leq A_{CP} = \sum_{k=1}^{n} f(v_k)\Delta x$, and $\lim_{n\to\infty} A_{IP} = \lim_{n\to\infty} A_{CP} = A$.

Approximating Area Using Partitions:

Given a function f on an interval $[a, b]$, we can also appoximate area using partitions that do not necessarily have rectangles all of the same width. A *partition* P of the interval $[a, b]$ of size n is a set of numbers $a = x_0 < x_1 < x_2 < ... < xn - 1 < x_n = b$. $\Delta x_k = x_k = x_{k-1}$ is the width of the kth subinterval, and $\|P\|$, the *norm* of the partition P, is the width of the widest of all the subintervals in P.

The *Riemann sum* of f on $[a, b]$ for a partition P is $R_P = \sum_{k=1}^{n} f(w_k)\Delta x$, where w_k is some point in the kth subinterval of the partition P.

If $\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(w_k)\Delta x = L$ for some real number L, then we say that f is integrable on $[a, b]$, and the definite integral of f on $[a, b]$ is:

$$\int_a^b f(x)dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(w_k)\Delta x = L$$

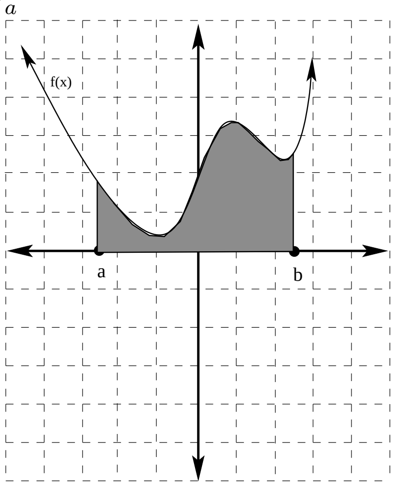
The Fundamental Theorem of Calculus:

Let f be a continuous function on the interval $[a, b]$:

(a) If G is the function defined by $\int_a^x f(t)\ dt$ for every x in $[a, b]$, then G is an antiderivative of f on $[a, b]$.

(b) If F is any antiderivative of f on $[a, b]$, then:

$$\int_a^b f(x)\ dx = F(b) - F(a)$$



Example: $\int_1^3 3x^2\ dx = x^3\ \Big|_1^3 = 3^3 - 1^3 = 27 - 1 = 26$

E. Properties of Definite Integrals

1. $\int_a^b c\ dx = c(b-a)$

2. $\int_a^a f(x)\ dx = 0$

3. $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$

4. $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$, for any constant c

5. $\int_a^b f(x) \pm g(x)\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$

6. $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$

   If f is integrable on $[a,b]$ and $f(x) \geq 0$ for every x in $[a,b]$, then $\int_a^b f(x)\ dx \geq 0$

   If f and g are integrable on $[a,b]$ and $f(x) \geq g(x)$ for every x in $[a,b]$, then $\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx$

   The Mean Value Theorem for Definite Integrals:

   If f is continuous on $[a,b]$, then there is a number z in the open interval (a,b) such that

   $\int_a^b f(x)\ dx = f(z)(b-a)$

F. The Average Value of a Function

Let f be a function that is integrable on an interval $[a,b]$. Then the **average value** of f over $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)\ dx$.