Binomial Experiments

Definition:
A sequence of binomial trials is called a **binomial experiment** and has the following properties:

- The experiment is performed for a fixed number of trials.
- The experiment can be thought of as having two outcomes, "success" and "failure".
- The probability of success is the same from trial to trial.
- The trials are independent of each other.

Examples and Non-Examples:

1. Flip a fair coin 5 times [Success: Heads, Failure: Tails, $P(S) = \frac{1}{2}$, number of trials: 5]
2. Roll one fair die 7 times [Success: 6 , Failure: 1 - 5, $P(S) = \frac{1}{5}$, number of trials: 7]
3. Roll 2 dice 4 times [Success: Roll < 7, Failure: Roll ≥ 7, $P(S) = \frac{15}{36}$, number of trials: 4]
4. Flip a coin until heads comes up [*not* a fixed number of trials.]
5. Select 2 cards, one at a time from a deck, without replacement [trials are *not* independent.]
6. Draw one chip out of a bag containing 3 Red chips, 4 green chips and 7 blue chips [trial has more than 2 outcomes.]

The Binomial Formula: $B(n, k; p) = C(n, k) \cdot (p)^k \cdot (1-p)^{n-k}$

This formula is used to compute the probability of obtaining exactly k successes after performing n binomial trials. That is, n is the *number of trials*, k is the number of successes, p is the probability of success, $(1-p)$ is the probability of failure, $n-k$ is the number of failures, and $C(n, k)$ is the number of different ways of choosing k successes out of the n trials performed.

Example: Suppose that you take a 10 question multiple choice test with exactly 4 choices for each answer. However, you did not study for the test and you know nothing about the subject, so you decide to answer each question by randomly selecting one of the four answers.
(a) Find the probability that you guess exactly 7 answers correctly.

Notice that this is $B\left(10, 7; \frac{1}{4}\right) = C(10, 7) \cdot \left(\frac{1}{4}\right)^7 \cdot \left(\frac{3}{4}\right)^3 = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} \cdot \left(\frac{1}{4}\right)^7 \cdot \left(\frac{3}{4}\right)^3 = \frac{19440}{6291456} \approx 0.003$

(b) Find the probability that you guess all 10 answers correctly.

Notice that this is $B\left(10, 10; \frac{1}{4}\right) = C(10, 10) \cdot \left(\frac{1}{4}\right)^{10} \cdot \left(\frac{3}{4}\right)^0 = 1 \cdot \left(\frac{1}{4}\right)^{10} = \frac{1}{4^{10}} \approx 0.000000954$

(c) Find the probability that you guess *at least 9* answers correctly.
Notice that since the binomial formula is used to compute the probability of getting *exactly* k correct, we will have to use the binomial formula more than once and add the results:

$$B\left(10, 9; \frac{1}{4}\right) = C(10, 9) \cdot \left(\frac{1}{4}\right)^9 \cdot \left(\frac{3}{4}\right)^1 = 10 \cdot \left(\frac{1}{4}\right)^9 \cdot \left(\frac{3}{4}\right)^1 = \frac{10 \cdot 3}{4^10}$$

$$B\left(10, 10; \frac{1}{4}\right) = C(10, 10) \cdot \left(\frac{1}{4}\right)^{10} \cdot \left(\frac{3}{4}\right)^0 = 1 \cdot \left(\frac{1}{4}\right)^{10} = \frac{1}{4^{10}}$$

Hence $P\left(\text{at least9correct}\right) = \frac{30}{4^{10}} + \frac{1}{4^{10}} = \frac{31}{4^{10}} \approx 0.000029564$