Math 102
Exam 3: Additional Practice Problem Solutions

1. Compute the value of each of the following:

   (a) $0! = 1$

   (b) $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

   (c) $\frac{10!}{7!} = 10 \cdot 9 \cdot 8 = 720$

   (d) $C(10, 7) = \frac{10 \cdot 9 \cdot 8}{3!} = 120$

2. A local restaurant has 4 appetizers, 12 entrees, and 5 desserts. Find the number of possible meals that can be formed by choosing exactly one menu item of each type.

   $(4)(12)(5) = 240$ possible meals

3. A company assigns billing codes to each of its clients consisting of two letters followed by three one-digit numbers. Find the number of possible billing codes if:

   (a) Repetition is allowed.
   $(26)(26)(10)(10)(10) = 676,000$ possible billing codes if repetition is allowed.

   (b) Repetition is **not** allowed.
   $(26)(25)(10)(9)(8) = 468,000$ possible billing codes if repetition is not allowed.

4. A club has 5 male members and 7 female members.

   (a) How many ways can a committee of 5 club members be chosen?
   Since there are 12 total members in the club, and no restrictions on who is on the committee, and the order people are put onto the committee does not matter, the number of possible ways of forming this committee is given by:
   $C(12, 5) = \frac{12!}{7!5!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 792$

   (b) How many ways can a committee of 5 be chosen if the committee is required to consist of 2 men and 3 women?
   Since we must have exactly 2 men and 3 women on the committee, and the order people are put onto the committee does not matter, the number of possible ways of forming this committee is given by:
   $C(5, 2) \cdot C(7, 3) = \frac{5!}{3!2!} \frac{7!}{4!3!} = \frac{5 \cdot 4}{2 \cdot 1} \cdot \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 350$

   (c) How many ways can a committee of 5 be chosen if one member is designated as the head of the committee, and the rest of the committee is required to consist of 2 men and 2 women?
   There are two ways to count this one. First, we can think about choosing the head of the committee first, in which case, we will need to consider two cases, the case when the head is male, and the case when the head is female. Since there are 5 men and 7 women in the club to choose from, and we want two of each to be on the committee in addition to the head, we have:
   $5 \cdot C(4, 2) \cdot C(7, 2) + 7 \cdot C(5, 2) \cdot C(6, 2) = 5 \cdot \frac{4!}{2!2!} \frac{7!}{5!2!} + 7 \cdot \frac{5!}{3!2!} \frac{6!}{4!2!} = 5 \cdot \frac{4 \cdot 3}{2 \cdot 1} \cdot \frac{7 \cdot 6}{2 \cdot 1} + 7 \cdot \frac{5 \cdot 4}{2 \cdot 1} \cdot \frac{6 \cdot 5}{2 \cdot 1} = 630 + 1050 = 1680$

   A slightly easier way to count this is to think of choosing the 2 male and 2 female committee members first, and then choosing the head from the remaining 8 club members. This gives:
   $C(5, 2) \cdot C(7, 2) \cdot 8 = \frac{5!}{3!2!} \frac{7!}{5!2!} \cdot 8 = \frac{5 \cdot 4}{2 \cdot 1} \cdot \frac{7 \cdot 6}{2 \cdot 1} \cdot 8 = 1680$

5. Suppose you go to the store and purchase a variety pack with 8 individually wrapped bags of chips, where each bag is of a different type. You plan to select one bag each day (Monday through Friday) to include as part of the lunch you take to work. How many different ways could the chips you bring to work for lunch that week be selected?

   Since we are bringing different bags of chips on different days, order matters, so this is a permutation problem. Therefore the count is given by: $P(8,5) = \frac{8!}{3!} = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 6720$

6. A bag contains 7 white chips, 3 red chips, and 2 blue chips.

   (a) Suppose 1 chip is randomly drawn from the bag.

   i. Find the probability that a blue chip is drawn.
   Since there are 12 total chips, 2 of which are blue, $P(B) = \frac{2}{12} = \frac{1}{6}$.

   ii. Find the *odds* in favor of drawing a white chip.
   Since there are 7 white chips, and 5 non-white chips, the odds in favor of drawing a white chip are $7 : 5$, or $\frac{7}{5}$ (either form is acceptable, although I prefer the first so that it looks different from a probability).

   (b) Now suppose that all of the 12 original chips have been returned to the bag, and then two chips are randomly drawn from the bag, one at a time, without replacement.

   i. Find the probability that both chips are red.
   Notice that at first, there are 12 total chips, 3 of which are red, but on the second draw, if we drew a red chip on the first draw, there are 11 chips, 2 of which are red. Therefore:
   $P(R, R) = \frac{3}{12} \cdot \frac{2}{11} = \frac{6}{132} = \frac{1}{22}$

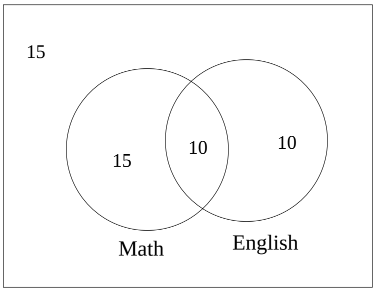
   ii. Find the probability that the first chip is white and the second chip is blue.
   This is similar to the previous part, except on the first draw, there are 12 chips, and we want to get one of the 7 white chips, while on the second draw, there are 11 chips, and we want to get one of the 2 blue ones. Therefore:
   $P(W, B) = \frac{7}{12} \cdot \frac{2}{11} = \frac{14}{132} = \frac{7}{66}$

   iii. Find the probability that neither chip is red.
   Note: This is **not** the same thing as finding the probability of the complement to part (i). Instead, we want to compute the probability of *avoiding* a red chip on both draws.
   Here, on the first draw, there are 12 chips, and we want to get one of the 9 non-red chips, while on the second draw, there are 11 chips, and we want to get one of the remaining 8 non-red ones. Therefore:
   $P(R', R') = \frac{9}{12} \cdot \frac{8}{11} = \frac{72}{132} = \frac{6}{11}$

   iv. Find the probability that **at least one** chip is blue.
   There are two ways to accomplish getting **at least one** blue chip. We could get a blue chip on the first draw, and any chip on the second draw, or we could get a non-blue chip on the first draw, and a blue chip on the second draw. Therefore:
   $P(\text{ at least one Blue}) = \frac{2}{12} \cdot \frac{11}{11} + \frac{10}{12} \cdot \frac{2}{11} = \frac{22}{132} + \frac{20}{132} = \frac{42}{132} = \frac{7}{22}$

7. A survey of 50 college students finds that 25 of them are taking Math this semester, 20 are taking English, and 10 are taking both Math and English. Suppose that a student is randomly selected from among the students who participated in the survey.

   (a) Find the probability that the student is taking Math but is not taking English.
   To solve this problem, is is helpful (but not required) to draw a Venn Diagram to illustrate this situation:

   From the diagram, we see that $P(M - E) = \frac{15}{50} = .30$

   (b) Find the probability that the student is taking either Math or English.
   $P(M \cup E) = P(M) + P(E) - P(M \cap E) = \frac{25}{50} + \frac{20}{50} - \frac{10}{50} = \frac{35}{50} = .70$

   (c) Find the probability that the student is taking neither Math nor English.
   $P((M \cup E)') = 1 - P(M \cup E) = \frac{50}{50} - \frac{35}{50} = \frac{15}{50} = .30$

   (d) Given the the student is taking Math, find the probability that the student is also taking English.
   $P(E|M) = \frac{P(E \cap M}{P(M)} = \frac{n(E \cap M)}{n(M)} = \frac{10}{25} = .40$

8. Consider the following game: A bag contains 10 red balls and 5 green balls. There are two ways to play -

   Option 1: Pay \$1 for the opportunity to draw one ball out of the bag. If you draw a red ball, you lose your \$1. If draw a green ball, you win \$3 (your original \$1, plus \$2 more).

   Option 2: Pay \$5 for the opportunity to draw two balls (without replacement) out of the bag. If the two balls you draw are different colors, you lose your \$5. If the two balls are the same color, you win \$10 (your original \$5, plus \$5 more).

   (a) Find the expected value for playing Option 1 of this game. Is this game fair?
   Notice that $P(\text{ Win }) = \frac{5}{15} = \frac{1}{3}$ while $P(\text{ Lose }) = \frac{10}{15} = \frac{2}{3}$.
   Since you gain \$2 if you win, and lose \$1 if you lose, the expected value for this game is given by:
   Exp. Value $= \left(\frac{1}{3}\right)(+\$2) + \left(\frac{2}{3}\right)(-\$1) = \frac{2}{3} - \frac{2}{3} = \$0$. Since the expected return is zero, neither the player nor the house has an advantage, so the game is fair.

   (b) Find the expected value for playing Option 2 of this game. Is this game fair?
   Computing the probability of winning in this situation involving two draws is a bit more work, but the idea is the same as before. We just need to use conditional probability to find the probability of the intersection of the events we are interested in.
   $P(\text{ Win }) = P(R,R) + P(G,G) = \frac{2}{3} \cdot \frac{9}{14} + \frac{1}{3} \cdot \frac{4}{14} = \frac{18}{42} + \frac{4}{42} = \frac{22}{42} = \frac{11}{21}$
   Similarly, $P(\text{ Lose }) = P(R,G) + P(G,R) = \frac{2}{3} \cdot \frac{5}{14} + \frac{1}{3} \cdot \frac{10}{14} = \frac{10}{42} + \frac{10}{42} = \frac{20}{42} = \frac{10}{21}$ [Or, we could just realize that $P(\text{ Lose }) = 1 - P(\text{ Win }) = \frac{10}{21}$]. Therefore:
   Exp. Value $= \left(\frac{11}{21}\right)(+\$5) + \left(\frac{10}{21}\right)(-\$5) = \frac{55}{21} - \frac{50}{21} = \frac{5}{21} \approx \$.238$. So the expected return is about 24 cents each time we play the game. Sine the player has an advantage, the game is not fair.

9. For a standard deck of 52 cards, find the probability of drawing 5 cards without replacement and getting a full house (three of a kind plus a pair).

   To find the probability of getting a full house, we need to count the number of full houses, and divide it by the total number of possible 5 card hands.

   The number of full houses can be found by first choosing one of the 13 types of card for our three of a kind and choosing three of these four cards, and then choosing one of the remaining 12 types of cards for our pair and choosing two of these four cards.

   Therefore, $P(\text{ Full House }) = \dfrac{13 \cdot C(4,3) \cdot 12C(4,2)}{C(52,5)} = \dfrac{\frac{13 \cdot 4! \cdot 12 \cdot 4!}{3!1!2!2!}}{\frac{52!}{47!5!}} = \dfrac{13 \cdot 4 \cdot 12 \cdot 6 \cdot 5!}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} = \dfrac{6}{4165} \approx .00144$

10. Suppose you are going to take a multiple choice test which has 6 questions, with each question having 4 options to choose from. Since you did not have time to study, you decide to answer by randomly guessing each answer.

    (a) Find the number of different ways one could complete this exam.

    Since each question has 4 possible answers, there are $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4096$ possible ways of completing this exam.

    (b) Find the probability of getting all 6 questions right by randomly guessing.

    Since the only way to get all 6 right is to "guess" all six correct answers, and there is only one way yo do this, the probability of getting all 6 questions correct is: $P(6 \text{ correct answers }) = \dfrac{1}{4096} \approx .000244141$ (not very good!)

    (c) Find the probability of getting **exactly** 5 questions right.

    Notice that we can think of getting 5 questions correct as picking one of the 6 to get wrong (there are 6 ways to do this) and then picking an incorrect answer for this question (there are 3 incorrect options to choose from). Therefore, there are 18 ways to get exactly 5 questions correct.

    Therefore, to find the probability of getting **exactly** 5 questions right, we compute:

    $P(5 \text{ correct}) = \dfrac{18}{4036} \approx .0043945$