Math 262
Exam 3 - Version 1 **Name:**______________________________________

Instructions: You will have 60 minutes to complete this exam. Calculators are allowed, but this is a closed book, closed notes exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit. Simplify answers when possible and follow directions carefully on each problem.

1. Evaluate each of the following integrals:

(a) (10 points) $\displaystyle\int \frac{1}{x^2+4x+20}\, dx$

We proceed by completing the square:

$= \displaystyle\int \frac{1}{(x^2+4x+4)+16}\, dx = \int \frac{1}{(x+2)^2+16}\, dx$

Let $u = x+2$. Then $du = dx$, so we have:

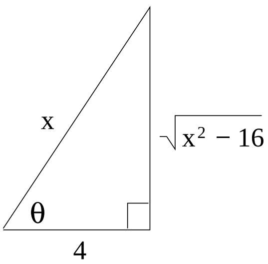
$\displaystyle\int \frac{1}{u^2+4^2}\, dx = \frac{1}{4}\arctan\left(\frac{u}{4}\right) + C = \frac{1}{4}\arctan\left(\frac{x+2}{4}\right) + C$

(b) (12 points) $\displaystyle\int \frac{\sqrt{x^2-16}}{2x}\, dx$

We proceed using trigonometric substitution: Let $x = 4\sec\theta$. Then $dx = 4\sec\theta\tan\theta$.

$= \displaystyle\int \frac{\sqrt{16\sec^2\theta - 16}}{2(4\sec\theta)} \cdot 4\sec\theta\tan\theta\, d\theta = \int \frac{4\tan\theta \cdot 4\sec\theta\tan\theta}{8\sec\theta}\, d\theta$

$= 2\displaystyle\int \tan^2\theta\, d\theta = 2\int \sec^2\theta - 1\, d\theta = 2\tan\theta - 2\theta + C$



$= \dfrac{\sqrt{x^2-16}}{2} - 2\text{arcsec}\dfrac{x}{4} + C$

(c) (12 points) $\displaystyle\int \frac{2x-1}{x^3+x}\, dx$

We proceed using partial fractions:

$= \displaystyle\int \frac{2x-1}{x(x^2+1)}\, dx = \int \frac{A}{x} + \frac{Bx+C}{x^2+1}\, dx.$

Therefore, $A(x^2+1) + x(Bx+C) = 2x-1$.

If $x = 0$, then we have $A = -1$.

Then we have $-x^2 - 1 + Bx^2 + Cx = 2x - 1$

Therefore, $-x^2 + Bx^2 = 0$, $Cx = 2x$, and $-1 = -1$.

Then $B = 1$ and $C = 2$, so we have:

$= \displaystyle\int -\frac{1}{x} + \frac{x+2}{x^2+1}\, dx = \int -\frac{1}{x} + \frac{x}{x^2+1} + \frac{2}{x^2+1}\, dx.$

$= -\ln|x| + \dfrac{1}{2}\ln(x^2+1) + 2\arctan x + C.$

(d) (12 points) $\displaystyle\int \frac{1}{2x+\sqrt[3]{x}}\,dx$

We proceed by using a non-standard substitution: Let $x = u^3$. Then $dx = 3u^2\,du$ and $u = \sqrt[3]{x}$. Then we have: $\displaystyle\int \frac{3u^2}{2u^3+u}\,du = 3\int \frac{u}{2u^2+1}\,du.$

Next, let $w = 2u^2+1$. Then $dw = 4u\,du$, or $\frac{1}{4}dw = u\,du$.

This gives: $\displaystyle\frac{3}{4}\int \frac{1}{w}\,dw = \frac{3}{4}\ln|w| + C = \frac{3}{4}|2u^2+1| + C = \frac{3}{4}|2x^{\frac{2}{3}}+1| + C$

2. Determine whether the following integrals converge or diverge. If they converge, evaluate the integral.

(a) (10 points) $\displaystyle\int_0^1 x\ln x\,dx$

Notice that this integrand is undefined when $x = 0$. Therefore, we consider:

$\displaystyle\lim_{t\to 0^+}\int_t^1 x\ln x\,dx$. Next, integrating by parts:

Let $u = \ln x$ and $dv = x$. Then $du = \frac{1}{x}dx$ and $v = \frac{1}{2}x^2$

Then we have: $\displaystyle = \lim_{t\to 0^+}\frac{1}{2}x^2\ln x\Big|_t^1 - \frac{1}{2}\int_t^1 x\,dx = \lim_{t\to 0^+}\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\Big|_t^1.$

$$= \lim_{t\to 0^+}\left[0-\frac{1}{4}\right] - \left[\frac{1}{2}t^2\ln t - \frac{1}{4}t^2\right]$$

Notice $\displaystyle\lim_{t\to 0^+}\frac{1}{2}t^2\ln t$ has the form $0\cdot\infty$, which is indeterminate, so we rewrite this as:

$$\lim_{t\to 0^+}\frac{\ln t}{t^{-2}}\left(\frac{0}{0}\right) = \lim_{t\to 0^+}\frac{\frac{1}{t}}{-2t^{-3}} = \lim_{t\to 0^+}\frac{t^3}{-2t} = \lim_{t\to 0^+}-\frac{1}{2}t^2 = 0$$

Then $\displaystyle\lim_{t\to 0^+}[0-\frac{1}{4}] - [\frac{1}{2}t^2\ln t - \frac{1}{4}] = -\frac{1}{4} - 0 = -\frac{1}{4}$

Therefore, this integral converges to $-\frac{1}{4}$

(b) (12 points) $\displaystyle\int_0^3 \frac{2x}{x^2-4}\,dx$

Notice that this integral is doubly improper (since $x \neq 2$). Therefore, we will look at:

$\displaystyle\lim_{t\to 2^-}\int_0^t \frac{2x}{x^2-4}\,dx + \lim_{t\to 2^+}\int_t^3 \frac{2x}{x^2-4}\,dx.$

$$= \lim_{t\to 2^-}\frac{1}{2}\ln|x^2-4|\Big|_0^t + \lim_{t\to 2^+}\frac{1}{2}|x^2-4|\Big|_t^3$$

$$= \lim_{t\to 2^-}\frac{1}{2}\ln|t^2-4| - \frac{1}{2}\ln|-4| + \lim_{t\to 2^+}\frac{1}{2}|3^2-4| - |t^2-4|, \text{ which diverges.}$$

3. (10 points) Use a comparison to determine whether the integral $\int_2^{\infty} \frac{3x+\sin^2 x}{x^3+1}\,dx$ converges or diverges.

First, we decide that we believe this integral converges, since the degree of the numerator is two more than that in the denominator. Therefore, we wish to compare our integrand to a larger function.

Notice that since $\sin^2 x \le 1$ $\frac{3x+\sin^2 x}{x^3+1} \le \frac{3x+1}{x^3+1} \le \frac{3x+1}{x^3} \le \frac{4x}{x^3} = \frac{4}{x^2}$

Next, notice that $\int_2^{\infty} \frac{4}{x^2}\,dx = \lim_{t\to\infty} 4\int_2^t x^{-2}\,dx = \lim_{t\to\infty} -x^{-1}\Big|_2^t$

$= \lim_{t\to\infty} 4\left[-\frac{1}{t}+\frac{1}{2}\right] = 2.$

Therefore, our original integral converges by comparison.

4. Evaluate the following limits:

(a) (8 points) $\lim_{x\to 0} \frac{3x - e^{3x} + \cos(x)}{x^2}$

Form: $\frac{0}{0}$, so by L'Hôpital's Rule: $= \lim_{x\to 0} \frac{3-3e^{3x}-\sin(x)}{2x}$

Form: $\frac{0}{0}$, so by L'Hôpital's Rule: $= \lim_{x\to 0} \frac{-9e^{3x}-\cos(x)}{2} = \frac{-10}{2} = -5.$

(b) (8 points) $\lim_{x\to 0^+} x\csc x$

Form: $0\cdot\infty$, so we must rewrite this before applying by L'Hôpital's Rule:

$= \lim_{x\to 0^+} \frac{x}{\sin x}.$

Form: $\frac{0}{0}$, so, applying by L'Hôpital's Rule:

$= \lim_{x\to 0^+} \frac{1}{\cos x} = 1.$

(c) (8 points) $\lim_{x\to 0^+} x^{\tan x}$

Let $y = x^{\tan x}$. Then $\ln y = \ln(x)^{\tan x} = \tan x \ln x$.

Notice that $\lim_{x\to 0^+} \tan x \ln x$ has the form $0\cdot\infty$, so we must rewrite this before applying by L'Hôpital's Rule:

$= \lim_{x\to 0^+} \frac{\ln x}{\cot x}.$

Form: $\frac{\infty}{\infty}$, so, applying by L'Hôpital's Rule:

$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\csc^2 x} = \lim_{x\to 0^+} \frac{-\sin^2 x}{x}.$

Form: $\frac{\infty}{\infty}$, so, applying by L'Hôpital's Rule:

$= \lim_{x\to 0^+} \frac{-2\sin x\cos x}{1} = 0.$

Hence $\lim_{x\to 0^+} x^{\tan x} = e^0 = 1$