Math 262
Exam 3
Practice Problem Solutions

1. Evaluate the following integrals:

(a) $\int \sec^3 x \tan^3 x \; dx$

$= \int tan^2 x \sec^2 x \cdot \sec x \tan x \; dx = \int (sec^2 x - 1) \sec^2 x \cdot \sec x \tan x \; dx$

Let $u = \sec x$. Then $du = \sec x \tan x \; dx$, and we have $\int (u^2 - 1)u^2 \; du = \int u^4 - u^2 \; du$

$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$

(b) $\int \frac{x^2}{x^2+9} \; dx$

This can be done using trig substitution, but here is an easier way:

$\int \frac{x^2}{x^2+9} \; dx = \int \frac{x^2+9}{x^2+9} - \frac{9}{x^2+9} \; dx \; dx$

$\int 1 - \frac{9}{x^2+9} \; dx = \int 1 \, dx - 9 \int \frac{1}{x^2+9} \; dx = x - 3 \arctan\left(\frac{x}{3}\right) + C$

(c) $\int \frac{x^2}{\sqrt{9-x^2}} \; dx$

Let $x = 3\sin\theta$. Then $dx = 3\cos\theta d\theta$, and we have $\int \frac{9\sin^2\theta}{\sqrt{9 - 9\sin^2\theta}} \cdot 3\cos\theta \; d\theta = \int \frac{27\sin^2\theta\cos\theta}{3\cos\theta} \; d\theta$

$= 9 \int \sin^2\theta \; d\theta = 9 \int \frac{1}{2} - \frac{1}{2}\cos(2\theta) \; d\theta = \frac{9}{2}\theta - \frac{9}{4}\sin(2\theta) + C = \frac{9}{2}\theta - \frac{9}{2}\sin(\theta)\cos(\theta) + C$

$= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{2} + C$

(d) $\int \frac{x^2}{\sqrt{x^2-9}} \; dx$

Let $x = 3\sec\theta$. Then $dx = 3\sec\theta\tan\theta \, d\theta$, and we have $\int \frac{9\sec^2\theta}{\sqrt{9\sec^2\theta - 9}} \cdot 3\sec\theta\tan\theta \; d\theta = \int \frac{27\sec^3\theta\tan\theta}{3\tan\theta} \; d\theta$

$= 9 \int \sec^3\theta \; d\theta = 9 \int \sec\theta \cdot \sec^2\theta \; d\theta$

Aside: Let $u = \sec\theta$ and $dv = \sec^2\theta d\theta$. Then $du = \sec\theta\tan\theta \, d\theta$ and $v = \tan\theta$.

Then $\int sec^3\theta \; d\theta = \sec\theta\tan\theta - \int \tan^2\theta\sec\theta \; d\theta = \sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta \; d\theta$

$= \sec\theta\tan\theta - \int \sec^3\theta - \sec\theta \; d\theta.$

Hence $2\int sec^3\theta \; d\theta = \sec\theta\tan\theta + \int \sec\theta \, d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C$

Therefore $\int sec^3\theta \; d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C'$

Thus our original integral is equal to:

$= \frac{9}{2}\sec\theta\tan\theta + \frac{9}{2}\ln|\sec\theta + \tan\theta| + C$

$= \frac{9}{2}\frac{x}{3}\frac{\sqrt{x^2-9}}{3} + \frac{9}{2}\ln|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}| + C = \frac{x\sqrt{x^2-9}}{2} + \frac{9}{2}\ln|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}| + C$

(e) $\int \frac{3x}{x^2 - 3x - 4} \; dx$

$= \int \frac{3x}{(x-4)(x+1)} \; dx = \int \frac{A}{x-4} + \frac{B}{x+1} \; dx$ where $A(x+1) + B(x-4) = 3x.$

Solving this, we get $A = \frac{12}{5}$ and $B = \frac{3}{5}$

Then $= \displaystyle\int \frac{\frac{12}{5}}{x-4} + \frac{\frac{3}{5}}{x+1}\ dx = \frac{12}{5}\ln|x-4| + \frac{3}{5}\ln|x+1| + C$

(f) $\displaystyle\int \frac{x^3+x+2}{x^2+2x-8}\ dx$

Using long division of polynomials, we see that $\frac{x^3+x+2}{x^2+2x-8} = x - 2 + \frac{13x-14}{x^2+2x-8}$

Therefore $= \displaystyle\int x - 2 + \frac{13x-14}{(x+4)(x-2)}\ dx = \int x - 2 + \frac{A}{x+4} + \frac{B}{x-2}\ dx$

where $A(x-2) + B(x+4) = 13x - 14$. Solving this, we see $A = 11$ and $B = 2$.

Thus $= \displaystyle\int x - 2 + \frac{11}{x+4} + \frac{2}{x-2}\ dx$

$= \frac{1}{2}x^2 - 2x + 11\ln|x+4| + 2\ln|x-2| + C$

(g) $\displaystyle\int \frac{3x+8}{x^3+5x^2+6x}\ dx$

$= \displaystyle\int \frac{3x+8}{x(x+3)(x+2)}\ dx = \int \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x+2}\ dx$

where $A(x+3)(x+2) + B(x)(x+2) + C(x)(x+3) = 3x + 8$

Solving this gives $A = \frac{4}{3}$, $B = -\frac{1}{3}$ and $C = -1$.

Therefore, we have $\displaystyle\int \frac{\frac{4}{3}}{x} - \frac{\frac{1}{3}}{x+3} - \frac{1}{x+2}\ dx$

$= \displaystyle\frac{4}{3}\ln|x| - \frac{1}{3}\ln|x+3| - \ln|x+2| + C$

(h) $\displaystyle\int \frac{x+2}{x^3+x}\ dx$

$= \displaystyle\int \frac{x+2}{x(x^2+1)}\ dx = \int \frac{Ax+B}{x^2+1} + \frac{C}{x}\ dx$

where $(Ax+B)x + C(x^2+1) = x + 2$. Solving this gives $A = -2$, $B = 1$ and $C = 2$.

Therefore, we have $\displaystyle\int \frac{-2x+1}{x^2+1} + \frac{2}{x}\ dx = \int \frac{-2x}{x^2+1} + \frac{1}{x^2+1} + \frac{2}{x}\ dx$

$= -\ln|x^2+1| + \arctan(x) + 2\ln|x| + C$

(i) $\displaystyle\int \frac{4}{x^2+2x+10}\ dx$

$= \displaystyle\int \frac{4}{x^2+2x+1+9}\ dx = 4\int \frac{1}{(x+1)^2+9}\ dx$

$= \displaystyle 4\left[\frac{1}{3}\arctan\left(\frac{x+1}{3}\right)\right] + C = \frac{4}{3}\arctan\left(\frac{x+1}{3}\right) + C$

(j) $\displaystyle\int \frac{4}{(x^2+2x+10)^{\frac{3}{2}}}\ dx$

$= \displaystyle 4\int \frac{1}{((x+1)^2+9)^{\frac{3}{2}}}\ dx$

Let $x + 1 = 3\tan\theta$. Then $dx = 3\sec^2\theta d\theta$, and we have:

$= \displaystyle 4\int \frac{3\sec^2\theta}{(9\tan^2\theta+9)^{\frac{3}{2}}}\ d\theta = 12\int \frac{\sec^2\theta}{(9\sec^2\theta)^{\frac{3}{2}}}\ d\theta = \frac{12}{27}\int \frac{\sec^2\theta}{\sec^3\theta}\ d\theta = \frac{12}{27}\int \cos\theta\ d\theta$

$= \displaystyle\frac{12}{27}\sin\theta + C = \frac{12}{27}\left(\frac{x+1}{\sqrt{x^2+2x+10}}\right) + C$

(k) $\displaystyle\int \frac{3x-1}{\sqrt{12-4x-x^2}}\ dx$

$= \displaystyle\int \frac{3x-1}{\sqrt{16-(4+4x+x^2)}}\ dx = \int \frac{3x-1}{\sqrt{16-(x+2)^2}}\ dx$

Let $u = x + 2$. Then $du = dx$ and $u - 2 = x$. Therefore, we have:

$$= \int \frac{3(u-2)-1}{\sqrt{16-u^2}}\ du = \int \frac{3u}{\sqrt{16-u^2}} - \frac{7}{\sqrt{16-u^2}}\ du$$
$$= -\frac{3}{2}(16-u^2)^{\frac{1}{2}} - 7\arcsin\left(\frac{u}{4}\right) + C = -\frac{3}{2}(12-4x-x^2)^{\frac{1}{2}} - 7\arcsin\left(\frac{x+2}{4}\right) + C$$

(l) $\int \frac{3x+5}{\sqrt{3x+1}}\ dx$

Let $u = 3x+1$. Then $\frac{1}{3}du = dx$ and $u+4 = 3x+5$, so we have:
$$= \frac{1}{3}\int \frac{u+4}{\sqrt{u}}\ du = \frac{1}{3}\int u^{\frac{1}{2}} + 4u^{-\frac{1}{2}}\ du$$
$$= \frac{1}{3}\left[\frac{2}{3}u^{\frac{3}{2}+8u^{\frac{1}{2}}}\right] + C = \frac{2}{9}(3x+1)^{\frac{3}{2}+\frac{8}{3}(3x+1)^{\frac{1}{2}}} + C$$

(m) $\int \frac{x^2}{(3x+4)^{10}}\ dx$

Let $u = 3x+4$. Then $\frac{1}{3}du = dx$ and $\frac{u-4}{3} = x$, so we have:
$$= \frac{1}{3}\int \frac{\left(\frac{u-4}{3}\right)^2}{u^10}\ du = \frac{1}{27}\int \frac{u^2-8u-16}{u^10}\ du = \frac{1}{27}\int u^{-8} - 8u^{-9} + 16u^{-10}\ du$$
$$= \frac{1}{27}\left[\frac{-1}{7}u^{-7} - 8\frac{-1}{8}u^{-8} + 16\frac{-1}{9}u^{-9}\right] + C = -\frac{1}{189}(3x+4)^{-7} + (3x+4)^{-8} - \frac{16}{9}(3x+4)^{-9} + C$$

(n) $\int \frac{1}{\sqrt[4]{x}+\sqrt[3]{x}}\ dx$

Let $x = u^12$. Then $dx = 12u^11$ and $x^{\frac{1}{12}} = u$. Therefore, we have:
$$= \int \frac{12u^11}{u^3+u^4}\ du = 12\int \frac{u^8}{1+u}\ du$$
Using long division, we obtain: $= 12\int u^7 - u^6 + u^5 - u^4 + u^3 - u^2 + u - 1 + \frac{1}{u+1}\ du$
$$= 12\left[\frac{1}{8}u^8 - \frac{1}{7}u^7 + \frac{1}{6}u^6 - \frac{1}{5}u^5 + \frac{1}{4}u^4 - \frac{1}{3}u^3 + \frac{1}{2}u^2 - u + \ln|u+1|\right] + C$$
$$= \tfrac{3}{2}x^{\frac{2}{3}} - \tfrac{12}{7}x^{\frac{7}{12}} + 2x^{\frac{1}{2}} - \tfrac{12}{5}x^{\frac{5}{12}} + 3x^{\frac{1}{3}} - 4x^{\frac{1}{4}} + 6x^{\frac{1}{6}} - x^{\frac{1}{12}} + \ln|x^{\frac{1}{12}}+1| + C$$

(o) $\int_0^1 x^{-\frac{1}{3}}\ dx$

$= \lim_{t\to 0^+}\left(\int_t^1 x^{-\frac{1}{3}}\ dx\right) = \lim_{t\to 0^+}\left(\frac{3}{2}x^{\frac{2}{3}}\Big|_t^1\right) = \lim_{t\to 0^+}\left(\frac{3}{2} - \frac{3}{2}t^{\frac{2}{3}}\right) = \frac{3}{2}$, so this integral converges to $\frac{3}{2}$.

(p) $\int_1^\infty x^{-\frac{1}{3}}\ dx$

$= \lim_{t\to\infty}\left(\int_1^t x^{-\frac{1}{3}}\ dx\right) = \lim_{t\to\infty}\left(\frac{3}{2}x^{\frac{2}{3}}\Big|_1^t\right) = \lim_{t\to\infty}\left(\frac{3}{2}t^{\frac{2}{3}} - \frac{3}{2}\right)$, which grows without bound, so this integral diverges.

(q) $\int_0^2 \frac{x}{\sqrt{4-x^2}}\ dx$

$$= \lim_{t\to 2^-}\left(\int_0^t \frac{x}{\sqrt{4-x^2}}\ dx\right)$$
Let $u = 4-x^2$. Then $du = -2x\,dx$ or $-\frac{1}{2}du = dx$. So we have:

$= \lim_{t\to 2^-}\left(-\frac{1}{2}\int_*^* \frac{1}{u^{\frac{1}{2}}}\ du\right) = \lim_{t\to 2^-}\left(-\frac{1}{2}[2u^{\frac{1}{2}}]\Big|_*^*\right) = \lim_{t\to 2^-}\left(-(4-x^2)^{\frac{1}{2}}\Big|_0^t\right)$

$= \lim_{t\to 2^-}\left(-(4-t^2)^{\frac{1}{2}} + \sqrt{4}\right) = -2$, so this integral converges to $-2$.

(r) $\int_0^2 \frac{1}{\sqrt{4-x^2}}\ dx$

$= \lim_{t\to 2^-} \left(\int_0^t \frac{1}{\sqrt{4-x^2}}\,dx\right) = \lim_{t\to 2^-} \left(\arcsin\left(\frac{x}{2}\right)\Big|_0^t\right) = \lim_{t\to 2^-} \left(\arcsin(\frac{t}{2}) - \arcsin(0)\right) = \arcsin(1) - \arcsin(0)$

$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$, so this integral converges to $\frac{\pi}{2}$.

(s) $\displaystyle\int_0^2 \frac{1}{4-x^2}\,dx$

$= \lim_{t\to 2^-} \left(\int_0^t \frac{1}{4-x^2}\,dx\right) = \lim_{t\to 2^-} \left(\int_0^t \frac{1}{(2+x)(2-x)}\,dx\right) = \lim_{t\to 2^-} \left(\int_0^t \frac{A}{2+x} + \frac{B}{2-x}\,dx\right)$, where:

$A(2-x) + B(2+x) = 1$. Solving this, we obtain $A = B = \frac{1}{4}$, so we then have:

$$= \lim_{t\to 2^-} \left(\frac{1}{4}\int_0^t \frac{1}{2+x} + \frac{1}{2-x}\,dx\right) = \lim_{t\to 2^-} \left(\frac{1}{4}\ln|2+x| + \frac{1}{4}\ln|2-x|\Big|_0^t\right)$$

$= \lim_{t\to 2^-} \left(\frac{1}{4}\ln|2+t| + \frac{1}{4}\ln|2-t| - \frac{1}{4}\ln|2| - \frac{1}{4}\ln|2|\right)$, which diverges.

(t) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt[3]{x}}\,dx$

Notice that this is triply improper, so we look at:

$$= \lim_{t\to\infty} \int_{-t}^{-1} \frac{1}{\sqrt[3]{x}}\,dx + \lim_{t\to 0^-} \int_{-1}^{t} \frac{1}{\sqrt[3]{x}}\,dx + \lim_{t\to 0^+} \int_{t}^{1} \frac{1}{\sqrt[3]{x}}\,dx + \lim_{t\to\infty} \int_{1}^{t} \frac{1}{\sqrt[3]{x}}\,dx.$$

$$= \lim_{t\to\infty} \left[\frac{3}{2}x^{\frac{2}{3}}\Big|_{-t}^{-1}\right] + \ldots$$

Notice that since this first piece diverges, we need not look at the others since we can already conclude that this integral diverges.

2. Find each limit, (if it exists).

(a) $\displaystyle\lim_{x\to 1} \frac{\sin(\pi x)}{x-1}$

Form: $\frac{0}{0}$, so, applying L'Hôpital's Rule:

$$\lim_{x\to 1} \frac{\sin(\pi x)}{x-1} = \lim_{x\to 1} \frac{\pi\cos(\pi x)}{1} = \frac{\pi(-1)}{1} = -\pi.$$

(b) $\displaystyle\lim_{x\to 1} \frac{e^{x-1}-1}{x^2-1}$

Form: $\frac{0}{0}$, so, applying L'Hôpital's Rule:

$$\lim_{x\to 1} \frac{e^{x-1}-1}{x^2-1} = \lim_{x\to 1} \frac{e^{x-1}}{2x} = \frac{e^0}{2} = \frac{1}{2}.$$

(c) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$

Form: $\frac{\infty}{\infty}$, so, applying L'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-\frac{1}{2}}} = \lim_{x\to\infty} \frac{2x^{\frac{1}{2}}}{x} = \lim_{x\to\infty} \frac{2}{x^{\frac{1}{2}}} = 0$$

(d) $\displaystyle\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$

Form: $\infty\cdot$, so, before applying L'Hôpital's Rule, we rewrite this as:

$\displaystyle\lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$, which has the form $\frac{0}{0}$, so, applying L'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{-x^{-2}\cos\left(\frac{1}{x}\right)}{-x^{-2}} = \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)}{1} = 1$$

(e) $\displaystyle\lim_{x\to 0} \frac{x\sin x}{\cos x - 1}$

Form: $\frac{0}{0}$, so, applying L'Hôpital's Rule:

$$\lim_{x\to 0}\frac{x\sin x}{\cos x - 1} = \lim_{x\to 0}\frac{\sin x + x\cos x}{-\sin x}$$

This still has the form: $\frac{0}{0}$, so, applying L'Hôpital's Rule again:

$$= \lim_{x\to 0}\frac{\cos x + \cos x - x\sin x}{-\cos x} = \frac{1+1+0}{-1} = -2$$

(f) $\lim_{x\to\infty}\left(\sqrt{x^2+1} - x\right)$

Form: $\infty - \infty$. This is a little tricky, since it is not a fractional form. The easiest way to transform this into a form where we can understand the limit is to rationalize the numerator of this expression:

$$\frac{\sqrt{x^2+1}-x}{1}\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} = \frac{x^2+1-x^2}{\sqrt{x^2+1}+x} = \frac{1}{\sqrt{x^2+1}+x}$$

Therefore, $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right) = \lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x} = 0$

(g) $\lim_{x\to 0}\frac{\sin x}{\cos x}$

Form: $\frac{0}{1}$. Thus $\lim_{x\to 0}\frac{\sin x}{\cos x} = 0$.

(h) $\lim_{x\to\infty}\left(\frac{1}{x}\right)^{\frac{1}{x}}$

Form: $0^0$, so we set $y = \left(\frac{1}{x}\right)^{\frac{1}{x}}$, so $\ln y = \frac{1}{x}\ln\left(\frac{1}{x}\right)$.

Form: $0\cdot\infty$, so we change the form into: $\frac{\ln\left(\frac{1}{x}\right)}{x}$, which has the form $\frac{\infty}{\infty}$, so applying L'Hôpital's Rule:

$$\lim_{x\to\infty}\frac{\ln\left(\frac{1}{x}\right)}{x} = \lim_{x\to\infty}\frac{\frac{1}{\frac{1}{x}}\cdot\frac{-1}{x^2}}{1} = \lim_{x\to\infty}\frac{-x}{x^2} = \lim_{x\to\infty}\frac{-1}{x} = 0.$$

Hence $\lim_{x\to\infty}\left(\frac{1}{x}\right)^{\frac{1}{x}} = e^0 = 1$

(i) $\lim_{x\to 0^+}(\cos x)^{\frac{1}{x}}$

Form: $1^\infty$, so we set $y = (\cos x)^{\frac{1}{x}}$, so $\ln y = \frac{1}{x}\ln(\cos x)$.

Form: $\infty\cdot 0$, so we change the form into: $\frac{\ln(\cos x)}{x}$, which has the form $\frac{0}{0}$, so applying L'Hôpital's Rule:

$$\lim_{x\to\infty}\frac{\ln(\cos x)}{x} = \lim_{x\to\infty}\frac{\frac{1}{\cos x}\cdot(-\sin x)}{1} = \lim_{x\to\infty}\frac{-\sin x}{\cos x} = 0.$$

Hence $\lim_{x\to 0^+}(\cos x)^{\frac{1}{x}} = e^0 = 1$

3. Use a comparison to determine whether the following integrals converge or diverge:

(a) $\int_1^\infty \frac{x}{1+x^3}\,dx$

Since the degree of the denominator exceeds that of the numerator by more than one, we expect that this integral will converge. To that end, we notice that:

$\frac{x}{1+x^3} \le \frac{x}{x^3} = \frac{1}{x^2}$ and so we look at:

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t x^{-2}\,dx = \lim_{t\to\infty} -\frac{1}{x}\Bigg|_1^t = \lim_{t\to\infty} -\frac{1}{t} + 1 = 1$$

Hence, $\int_1^\infty \frac{x}{1+x^3}\,dx$ converges by comparison.

(b) $\int \frac{2+\sin x}{\sqrt{x}}\,dx$

Since $\sin x$ is bounded ($-1 \le \sin x \le 1$), and $\sqrt{x}$ has exponent less than 1, we expect this integral to diverge. To that end, we notice that:

$\frac{2+\sin x}{x^{\frac{1}{2}}} \ge \frac{1}{x^{\frac{1}{2}}} = x^{-\frac{1}{2}}$ and so we look at:

$$\int \frac{1}{\sqrt{x}}\ dx = \lim_{t\to\infty}\int_1^t x^{-\frac{1}{2}}\ dx = \lim_{t\to\infty} 2x^{\frac{1}{2}}\Big|_1^t = \lim_{t\to\infty} 2\sqrt{t}-2\sqrt{1}, \text{ which diverges.}$$

Hence, $\displaystyle\int \frac{2+\sin x}{\sqrt{x}}\ dx$ diverges by comparison.

(c) $\displaystyle\int_2^\infty \frac{x}{x^{\frac{3}{2}}-1}\ dx$

We once again have a function where the degree of the denominator only exceeds the degree of the numerator by $\frac{1}{2}$, so we expect that this integral will diverge. To that end, we notice that:

$\frac{x}{x^{\frac{3}{2}}-1} \geq \frac{x}{x^{\frac{3}{2}}} = \frac{1}{x^{\frac{1}{2}}}$ and so we look at:

$$\int \frac{1}{\sqrt{x}}\ dx = \lim_{t\to\infty}\int_2^t x^{-\frac{1}{2}}\ dx = \lim_{t\to\infty} 2x^{\frac{1}{2}}\Big|_2^t = \lim_{t\to\infty} 2\sqrt{t}-2\sqrt{2}, \text{ which diverges.}$$

Hence, $\displaystyle\int \frac{x}{x^{\frac{3}{2}}-1}\ dx$ diverges by comparison.

(d) $\displaystyle\int_0^\infty \frac{\sin^2 x}{1+e^x}\ dx$

Since $0 \leq \sin^2 x \leq 1$ and $e^x$ grows without bound as $x \to \infty$, we expect that this integral converges. To that end, we notice that:

$\displaystyle\frac{\sin^2 x}{1+e^x} \leq \frac{1}{e^x}$ and so we look at:

$$\int \frac{1}{e^x}\ dx = \lim_{t\to\infty}\int_0^t e^{-x}\ dx = \lim_{t\to\infty} -e^{-x}\Big|_0^t = \lim_{t\to\infty} -e^{-t}+1, \text{ which converges.}$$

Hence, $\displaystyle\int_0^\infty \frac{\sin^2 x}{1+e^x}\ dx$ converges by comparison.