Math 127
Exam 4 Practice Problem Solutions

1. Translate each of the following expressions into exponential form:
   (a) $\log_5 x = y$
   $5^y = x$
   (b) $\log_z 5 = y$
   $z^y = 5$
   (c) $\log_y x = 5$
   $y^5 = x$

2. Find the *exact* value of each of the following:
   (a) $\log_{21}(1) = 0$ since $21^0 = 1$
   (b) $\log_7(0)$ is undefined since there is no exponent $x$ such that $7^x = 0$.
   (c) $\log_2(\frac{1}{8}) = -3$, since $2^{-3} = \frac{1}{8}$.
   (d) $\log_3(27) = 3$, since $3^3 = 27$
   (e) $\ln(e^2) = 2$ since ln represents $\log_e$.
   (f) $\log .0001 = -4$ since $10^{-4} = .0001$
   (g) $\log_9(27) = 1.5$, since $27 = 3^3 = \left(9^{\frac{1}{2}}\right)^3 = 9^{\frac{3}{2}}$.
   (h) $7^{\log_7(\pi)} = \pi$, by the inverse property of logarithms and exponentials.

3. Determine whether the following are True or False:
   (a) $\ln\left(\dfrac{x^3}{(x+1)(x-1)}\right) = 3\ln x - \ln(x+1) + \ln(x-1)$
   **False.** Notice that $\ln\left(\dfrac{x^3}{(x+1)(x-1)}\right) = 3\ln x - \ln(x+1) - \ln(x-1)$.

   (b) $e^{\ln\left(x^2+1\right)} = x^2 + 1$
   **True.** This is the inverse function property of exponential and log functions.

   (c) $e^{x^2} \cdot e^{3x} = e^{3x^3}$
   **False.** In fact, $e^{x^2} \cdot e^{3x} = e^{x^2+3x}$. The exponents **add** rather than multiply here.

   (d) $\dfrac{\ln(4x)}{\ln(2x)} = \ln 2$
   **False.** This is not a legal simplification. For example, if $x = 1$, then $\dfrac{\ln(4x)}{\ln(2x)} = \dfrac{\ln(4)}{\ln(2)} = 2$, while $\ln 2 \approx .6931$.

   (e) $\ln\left(e^{x^2} - 4\right) = x^2 - \ln 4$
   **False.** The inverse function property does not apply here because of the subtracton operation. In fact, if $x = 2$, $\ln\left(e^{2^2} - 4\right) \approx 3.924$, while $2^2 - \ln 4 \approx 2.614$

4. Use properties of logarithms to expand the following expression:
   $\log\left(\dfrac{x^4 z^2}{\sqrt[3]{y}}\right)$
   $= \log(x^4 z^2) - \log(\sqrt[3]{y}) = \log(x^4) + \log(z^2) - \log(y^{\frac{1}{3}})$
   $= 4\log x + 2\log z - \frac{1}{3}\log y$

5. Use the laws of logarithms to expand the expression: $\ln\left(\dfrac{x^2(x-1)^{\frac{5}{2}}}{(x-4)^3}\right)$

$$\ln\left(\frac{x^2(x-1)^{\frac{5}{2}}}{(x-4)^3}\right) = \ln x^2 + \ln(x-1)^{\frac{5}{2}} - \ln(x-4)^3 = 2\ln x + \frac{5}{2}\ln(x-1) - 3\ln(x-4)$$

6. Use the properties of logarithms to write the following as a single logarithm:

$\dfrac{3}{2}\log_b x^3y^4 - \dfrac{2}{3}\log_b x^4y^3 - 2\log_b xy$

$= \log_b \left(x^3y^4\right)^{\frac{3}{2}} - \log_b \left(x^4y^3\right)^{\frac{2}{3}} - \log_b(xy)^2$

$= \log_b \left(x^{\frac{9}{2}}y^6\right) - \left[\log_b \left(x^{\frac{8}{3}}y^2\right) + \log_b(x^2y^2)\right]$

$= \log_b \left(x^{\frac{9}{2}}y^6\right) - \log_b \left(x^{\frac{8}{3}}y^2x^2y^2\right)$

$= \log_b \left(x^{\frac{9}{2}}y^6\right) - \log_b \left(x^{\frac{14}{3}}y^4\right)$

$= \log_b \dfrac{x^{\frac{9}{2}}y^6}{x^{\frac{14}{3}}y^4} = \log_b x^{-\frac{1}{6}}y^2$

7. Use the change of base formula to approximate the following:

   (a) $\log_5 10$

   By the change of base formula: $\log_b u = \dfrac{\log_a u}{\log_a b}$

   Thus $\log_5 10 = \dfrac{\ln 10}{\ln 5} \approx 1.4307$

   (b) $\log_9 12$

   By the change of base formula: $\log_b u = \dfrac{\log_a u}{\log_a b}$

   Thus $\log_9 12 = \dfrac{\ln 12}{\ln 9} \approx 1.1309$

   (c) $\log_{15} 7$

   By the change of base formula: $\log_b u = \dfrac{\log_a u}{\log_a b}$

   Thus $\log_{15} 7 = \dfrac{\ln 7}{\ln 15} \approx 0.7186$

8. Convert the following functions to base $e$ exponentials:

   (a) $f(x) = 100 \cdot 4^x$

   $f(x) = 100 \cdot 4^x = 100e^{(\ln 4)x} \approx 100e^{1.3863x}$

   (b) $g(x) = 50 \cdot \left(\frac{1}{4}\right)^x$

   $g(x) = 50 \cdot \left(\frac{1}{4}\right)^x = 50e^{\ln\left(\frac{1}{4}\right)x} \approx 50e^{-1.3863x}$

   (c) $h(x) = 450 \cdot 5^{-2x}$

   $h(x) = 450 \cdot 5^{-2x} = 450 \cdot \left(\frac{1}{25}\right)^x = 450e^{\ln\left(\frac{1}{25}\right)x} = 450e^{-3.2189x}$

9. Solve the following equations (give exact answers whenever possible):

   (a) $e^{3x-2} = e^{4-5x}$

   By the one-to-one property of exponentials, $3x - 2 = 4 - 5x$.
   Then $8x = 6$, so $x = \frac{6}{8}$ or $x = \frac{3}{4}$

(b) $9^{2x} = 27(3)^{2x+1}$
Notice that all the terms can be written as powers of 3:
$\left(3^2\right)^{2x} = 3^3(3)^{2x+1}$, or $3^{4x} = 3^{2x+4}$.
Therefore, $4x = 2x + 4$, so $2x = 4$, or $x = 2$.

(c) $\log_2(3x^2 - 3) = \log_2(x^2 + x)$
Since the base on both logarithms are the same, by the one-to-one property, $3x^2 - 3 = x^2 + x$.
Moving everything to one side, we have: $2x^2 - x - 3 = 0$, or $(2x - 3)(x + 1) = 0$
Therefore, either $2x = 3$ or $x = -1$.
That is, either $x = \frac{3}{2}$, or $x = -1$.
However, notice that $x = -1$ does not check since you cannot take the logarithm of zero.
Thus the only solution is $x = \frac{3}{2}$.

(d) $\log_5(x^2 + 21) = 2$
Rewriting in exponential form, we have $5^2 = x^2 + 21$, or $25 = x^2 + 21$.
Therefore, $x^2 - 4 = 0$, or $(x + 2)(x - 2) = 0$. Thus either $x = 2$, or $x = -2$.
Notice that both of these solutions check, since $x^2 + 21$ is positive when $x = \pm 2$.

(e) $\log_2(2x) + \log_2(x - 3) = 3$
Using the properties of logarithms to write this as a single logarithm, we have:
$\log_2((2x)(x - 3)) = 3$, or $\log_2(2x^2 - 6x) = 3$.
Changing this to exponential form, we have $2^3 = 2x^2 - 6x$.
Moving everything to one side gives: $2x^2 - 6x - 8 = 0$, or $(2x - 8)(x + 1) = 0$. Thus either $2x = 8$, so $x = 4$, or $x = -1$.
However, notice that $x = -1$ does not check since you cannot take the logarithm of a negative quantity.
Thus the only solution is $x = 4$.

(f) $\log(\sqrt[4]{x+1}) = \dfrac{1}{2}$
Re-writing in exponential form gives us: $10^{\frac{1}{2}} = \sqrt[4]{x+1}$
Then $\left(10^{\frac{1}{2}}\right)^4 = \left(\sqrt[4]{x+1}\right)^4$, or $10^2 = x + 1$

Therefore $100 = x + 1$ or $x = 99$. (Notice this this solution does check since $\sqrt[4]{99+1} = \sqrt[4]{100} = 100^{\frac{1}{2}=10^{\frac{1}{2}}}$, and $\log 10^{\frac{1}{2}} = \frac{1}{2}$).

(g) $e^{2x-1} = 3$
Taking the natural log of both sides: $\ln\left(e^{2x-1}\right) = \ln 3$, or $2x - 1 = \ln 3$.
Therefore, $2x = \ln 3 + 1$, so $x = \frac{\ln 3 + 1}{2}$

(h) $4^{2x-1} = 3^{5x}$
Since the bases are different , we must solve this by taking the logarithm of both sides:
$\ln\left(4^{2x-1}\right) = \ln\left(3^{5x}\right)$.
Therefore, $(2x - 1)\ln 4 = 5x \ln 3$, or $(2\ln 4)x - \ln 4 = (5\ln 3)x$.
Getting every term containing an $x$ to one side and the constants on the other side:
$(2\ln 4)x - (5\ln 3)x = \ln 4$, or $x(2\ln 4 - 5\ln 3) = \ln 4$
Hence $x = \frac{\ln 4}{2\ln 4 - 5\ln 3} \approx -.5096$.

10. (a) Suppose you invest \$10,000 in a savings account that pays 3% annual interest compounded monthly. How much money will be in the account after 6 years?

$A = P(1 + \frac{r}{n})^{nt} = 10,000(1 + \frac{.03}{12})^{(12)(6)} = 10,000(1.0025)^{72} \approx \$11,969.48$

(b) How long would it take \$5,000 invested at 6% annual interest compounded continuously to triple?

The continuous interest formula is: $A = Pe^{rt}$. So we have $15,000 = 5,000e^{.06t}$, or $3 = e^{.06t}$, which makes sense since we want our initial investment to triple.
Taking the natural logarithm of both sides gives: $\ln 3 = \ln(e^{.06t}) = .06t$, so $t = \frac{\ln 3}{.06} \approx 18.31$ years.

(c) Find the interest rate needed for an investment of \$2,000 to double in 6 years if the interest is compounded quarterly.

Using the compound interest formula $A = P(1+\frac{r}{n})^{nt}$, we have $4,000 = 2,000(1+\frac{r}{4})^{(4)(6)}$, or $2 = (1+\frac{r}{4})^{24}$. Taking the natural log of both sides, $\ln 2 = \ln(1+\frac{r}{4})^{24} = 24\ln(1+\frac{r}{4})$, so $\frac{\ln 2}{24} = \ln(1+\frac{r}{4})$. Exponentiating both sides, we than have $e^{\frac{\ln 2}{24}} = e^{\ln(1+\frac{r}{4})} = 1+\frac{r}{4}$.

Hence $e^{\frac{\ln 2}{24}} - 1 = \frac{r}{4}$, therefore $4(e^{\frac{\ln 2}{24}} - 1) = r$, so $r \approx .1172$, or %11.72

11. Suppose that a culture of bacteria that initially has 500 cells grows to 10,000 cells in 12 hours.

(a) Find a function $f(t)$ that gives the number of cells in the culture as a function of time (in hours), assuming that this population grows continuously and exponentially.

Using the continuous exponential growth equation, $A = Pe^{rt}$, we see that $A = 10,000$, $p = 500$, and $t = 12$, or $10,000 = 500e^{12r}$. Therefore, $20 = e^{12r}$, so $\ln(20) = 12r$, and hence $r \approx .2496$.
Therefore, our function modeling the growth of this bacterial culture is: $f(t) = 500e^{.2496t}$.

(b) How long will it take for the culture to reach 1,000,000 cells?
Using the function $f(t) = 500e^{.2496t}$ found above, we solve $1,000,000 = 500e^{.2496t}$ for $t$.
Then $2000 = e^{.2496t}$, so $\ln(2000) = .2496t$, so $t = \frac{\ln(2000)}{.2496} \approx 30.45$ hours

12. Suppose a certain substance has a half life of 47 years. If you start with 100 grams of the substance, how long will it take for the amount to be reduced to 50 grams? How long will it take for the amount to be reduced to 12 grams?

First notice that since we start with 100 grams of the substance and the half-life is 47 years, it will take 47 years for the initial 100 grams to be reduced down to 50 grams.

Next, to find the time needed for the initial 100 grams to be reduced down to 12 grams, we will need to construct a model. Let $f(t) = A_0e^{kt}$ be our exponential decay model with $t$ in years. Then we have:
$\frac{1}{2}A_0 = A_0e^{k\cdot 47}$, or $\frac{1}{2} = e^{47k}$.
Then $\ln\left(\frac{1}{2}\right) = \ln\left(e^{47k}\right)$, or $\ln\left(\frac{1}{2}\right) = 47k$.
Thus $k = \frac{\ln\left(\frac{1}{2}\right)}{47} \approx -0.01475$.

From this, $f(t) = A_0e^{-0.01475t}$, or, starting with 100 grams and ending with 12 grams:
$12 = 100e^{-0.01475t}$, or $\frac{12}{100} = e^{-0.01475t}$. Thus $\ln\left(\frac{12}{100}\right) = -0.01475t$.
Hence $t = \frac{\ln\left(\frac{12}{100}\right)}{-0.01475} \approx 143.75$.
Therefore, it would take approximately 143.75 years for the initial 100 grams to be reduced down to 12 grams.

13. For each quadratic function given below, find the coordinates of the vertex, and find the equation for the line of symmetry for the graph of the function.

(a) $f(x) = (x-5)^2 + 12$

Since this quadratic function is already in vertex form, we can see from the form of the equation that the vertex is $(5, 12)$ and the axis of symmetry is at $x = 5$.

(b) $f(x) = (x+3)^2 - 8$

Since this quadratic function is already in vertex form, we can see from the form of the equation that the vertex is $(-3, -8)$ and the axis of symmetry is at $x = -3$.

(c) $f(x) = 2x^2 - 12x + 22$

This quadratic function is not in vertex form, so we can either use completing the square to put it in vertex form, or, instead, we can just recall that $h = \frac{-b}{2a} = \frac{12}{4} = 3$. Then $k = f(h) = f(3) = 2(3)^2 - 12(3) + 22 = 18 - 36 + 22 = 4$. Then the vertex is $(3, 4)$ and the axis of symmetry is at $x = 3$.

(d) $f(x) = -4x^2 + 16x - 13$

This quadratic function is not in vertex form, so we can either use completing the square to put it in vertex form, or, instead, we can just recall that $h = \frac{-b}{2a} = \frac{-16}{-8} = 2$. Then $k = f(h) = f(2) = -4(2)^2 + 16(2) - 13 = -16 + 32 - 13 = 3$. Then the vertex is $(2, 3)$ and the axis of symmetry is at $x = 2$.

14. For each quadratic function given below, find the vertex, line of symmetry, and (if applicable) the $x$-intercepts of the function. Then graph the function carefully in the $xy$-plane.

(a) $f(x) = x^2 - 4x$

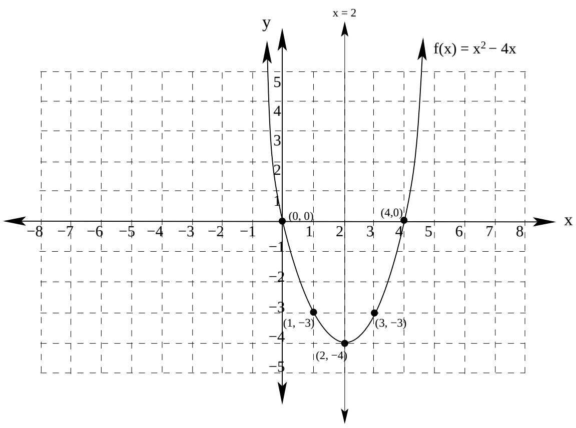
Since the form of this quadratic is fairly simple, we will go ahead and complete the square to put it into vertex form:

$f(x) = (x^2 - 4x + 4) - 4 = (x - 2)^2 - 4)$. So the vertex is $V : (2, -4)$ and the axis of symmetry is $x = 2$.

Next, we find the intercepts: $f(0) = 0$, so $(0, 0)$ is the $y$-intercept [notice that this is also an $x$-intercept]

To find the $x$-intercepts, we factor $x^2 - 4x = 0$ to get $x(x - 4) = 0$, so the intercepts are at $x = 0$ and $x = 4$.

Finally, we plot one more value to get a better idea of the shape of our graph: $f(1) = 1 - 4 = -3$, so $(1, -3)$ is on the graph, so using symmetry, $(3, -3)$ is also on the graph. Then we have the graph:




(b) $f(x) = 6x^2 + 7x - 24$

Since the form of this quadratic is a bit messy, we will find the vertex algebraically:

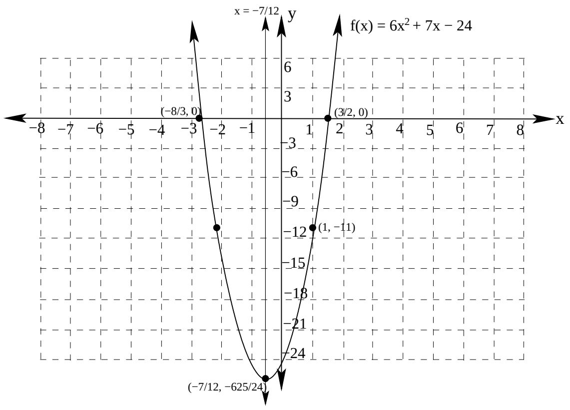
$h = \frac{-b}{2a} = -\frac{7}{12}$, so $k = f(h) = f\left(-\frac{7}{12}\right) = 6\left(-\frac{7}{12}\right)^2 + 7\left(-\frac{7}{12}\right) - 24 == -\frac{625}{24} \approx -26.042$

So the vertex is $V : (-\frac{7}{12}, -\frac{625}{24})$ and the axis of symmetry is $x = -\frac{7}{12}$

Next, we find the intercepts: $f(0) = -24$, so $(0, -24)$ is the $y$-intercept.

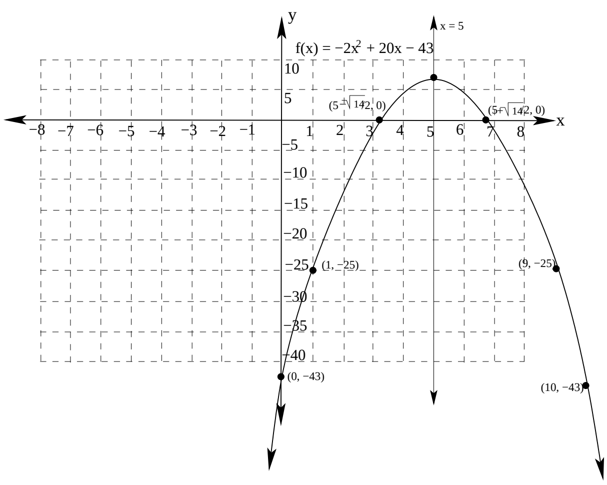
To find the $x$-intercepts, we factor $6x^2 + 7x - 24 = 0$ to get $(3x + 8)(2x - 3) = 0$, so the intercepts are at $x = \frac{3}{2}$ and $x = -\frac{8}{3}$.

Finally, we plot one more value to get a better idea of the shape of our graph: $f(1) = 6 + 7 - 24 = -11$, so $(1, -11)$ is on the graph.




(c) $f(x) = -2x^2 + 20x - 43$

Since the form of this quadratic is a bit messy, we will find the vertex algebraically:
$h = \frac{-b}{2a} = \frac{-20}{-4} = 5$, so $k = f(h) = f(5) = -2(5)^2 + 20(5) - 43 = -50 + 100 - 43 = 7$
So the vertex is $V : (5, 7)$ and the axis of symmetry is $x = 5$
Next, we find the intercepts: $f(0) = -43$, so $(0, -43)$ is the $y$-intercept.
To find the $x$-intercepts, we look at $-2x^2 + 20x - 43 = 0$. Since this does not factor, we use the quadratic equation to find $x = \frac{-20 \pm \sqrt{20^2 - 4(-2)(-43)}}{2(-2)} = \frac{-20 \pm \sqrt{400 - 344}}{-4} = 5 \pm \frac{\sqrt{56}}{-4} = 5 \pm \frac{\sqrt{14}}{2}$
Finally, we plot one more value to get a better idea of the shape of our graph: $f(1) = -2 + 20 - 43 = -25$, so $(1, -25)$ is on the graph.




15. For each polynomial function given below, use the degree of the polynomial and the sign of the leading coefficient to describe the end behavior of the polynomial.

(a) $f(x) = 5x^4 - 3x^2 + 7$

Since $5 > 0$ and the leading term $5x^4$ has even degree, both ends of this graph are moving in the positive direction.

(b) $f(x) = -2x^3 + 4x^2 - 15x + 12$

Since $-2 < 0$ and the leading term $-2x^3$ has odd degree, the left end of this graph is moving in the positive direction while the right end is moving in the negative direction.

(c) $f(x) = 12x^5 - 5x^4 + 3x^2 - 12x + 7$

Since $12 > 0$ and the leading term $12x^5$ has odd degree, the left end of this graph is moving in the negative direction while the right end is moving in the positive direction.

(d) $f(x) = -7x^6 + 5x^4 - 13x^3 + 21x^2 + 15x + 23$

Since $-7 < 0$ and the leading term $-7x^4$ has even degree, both ends of this graph are moving in the negative direction.

16. For each polynomial function given below, find the zeroes of the polynomial and also give the multiplicity of each zero.

(a) $f(x) = x^4 + 3x^3 - 4x^2$

Factoring, we have $f(x) = x^2(x^2 - 3x - 4) = x^2(x + 4)(x - 1)$.
Therefore, the zeroes are $x = 0$, which has multiplicity 2, $x = -4$, which has multiplicity 1, and $x = 1$, which has multiplicity 1.

(b) $f(x) = x^3 + 2x^2 - 4x - 8$

Factoring by grouping, we have $f(x) = x^2(x+2)-4(x+2) = (x^2-4)(x+2) = (x+2)(x-2)(x+2) = (x+2)^2(x-2)$.
Therefore, the zeroes are $x = -2$, which has multiplicity 2 and $x = 2$, which has multiplicity 1.

(c) $f(x) = x(x - 4)^4(x + 3)^3$

Since this expression has already been factored, we can just read off the zeroes and their multiplicities:
$x = 0$, which has multiplicity 1, $x = 4$, which has multiplicity 4, and $x = -3$, which has multiplicity 3.

(d) $f(x) = x^3(x - 2)^2(x^2 - 4)^5$

This expression has been partially factored, be we cannot just read off the zeroes in this form, we we factor further to get $f(x) = x^3(x - 2)^2(x^2 - 4)^5 = x^2(x - 2)^2((x + 2)(x - 2))^5 = x^2(x - 2)^2(x + 2)^5(x - 2)^5 = x^2(x - 2)^7(x + 2)^5$
Then the zeroes are: $x = 0$, which has multiplicity 3, $x = 2$, which has multiplicity 7, and $x = -2$, which has multiplicity 5.

17. Use the intermediate value theorem to show that $f(x) = 2x^3 + 5x^2 - 3$ has a zero between $x = -3$ and $x = -2$

Notice that $f(-3) = 2(-3)^3 + 5(-3)^2 - 3 = -54 + 45 - 3 = -12$

While $f(-2) = 2(-2)^3 + 5(-2)^2 - 3 = -16 + 20 - 3 = 1$

Since $f(-3)$ is negative and $f(-2)$ is positive, by the Intermediate Value Theorem, there must be a value $c$ with $-3 < c < -2$ and $f(c) = 0$.