

1. Determine whether or not each of the following functions is one-to-one.

(a) $f(x) = \ln(x^2)$

Notice that $f(2) = f(-2) = \ln 4$. Therefore, this function is not one-to-one.

(b) $f(x) = x^5 + 2x^3 - 2$

First notice that f is continuous. Next, $f'(x) = 5x^4 + 6x^2$, which is always positive. Therefore, f is a continuous function which is increasing throughout its domain. Thus $f(x)$ is a one-to-one function.

2. Determine whether or not each of the following functions is one-to-one. If it is, find the inverse function.

(a) $f(x) = x^3 + 4$

First notice that f is continuous. Next, $f'(x) = 3x^2$, which is always non-negative. Therefore, f is a continuous function which is increasing throughout its domain. Thus $f(x)$ is a one-to-one function.

Then, solving $y = x^3 + 4$, $y - 4 = x^3$, or $\sqrt[3]{y-4} = x$. Thus $f^{-1}(x) = \sqrt[3]{x-4}$.

Check: $f(\sqrt[3]{x-4}) = (\sqrt[3]{x-4})^3 + 4 = x - 4 + 4 = x$ and $f^{-1}(x^3 + 4) = \sqrt[3]{(x^3 + 4) - 4} = \sqrt[3]{x^3} = x$.

(b) $f(x) = x^4 - 2$

Notice that $f(1) = f(-1) = -1$. Therefore, f is not a one-to-one function on its full domain.

(c) $f(x) = \frac{x+3}{2x-1}$

Suppose that $f(a) = f(b)$ for some pair of real numbers $a, b (\neq \frac{1}{2})$. Then $\frac{a+3}{2a-1} = \frac{b+3}{2b-1}$.

Then $(a+3)(2b-1) = (b+3)(2a-1)$, so $2ab + 6b - a - 3 = 2ab + 6a - b - 3$.

Therefore, $7b = 7a$, or $b = a$. Hence $f(x)$ is one-to-one.

To find the inverse of $f(x)$, we solve $y = \frac{x+3}{2x-1}$ for x .

Then $y(2x-1) = x+3$, or $2xy - y = x+3$. Thus $2xy - x = y+3$, or $x(y-1) = y+3$

Hence $x = \frac{y+3}{2y-1}$, so $f^{-1}(x) = \frac{x+3}{2x-1}$

Check: $f\left(\frac{x+3}{2x-1}\right) = \frac{\frac{x+3}{2x-1} + 3}{2\left(\frac{x+3}{2x-1}\right) - 1} = \frac{\frac{(x+3)+3(2x-1)}{2x-1}}{\frac{2(x+3)-(2x-1)}{2x-1}} = \frac{7x}{2x-1} \cdot \frac{2x-1}{7} = x$

3. (4 points each) Given the following table of values and that $g(x) = f^{-1}(x)$:

	$f(x)$	$f'(x)$
0	4	-1
1	2	3
2	1	5
3	-5	7
4	3	-2

Find the following:

(a) $g(2) = f^{-1}(2) = 1$

(b) $g(4) = f^{-1}(4) = 0$

(c) $g'(4) = \frac{1}{f'(g(4))} = \frac{1}{f'(0)} = -1$

(d) $g'(3) = \frac{1}{f'(g(3))} = \frac{1}{f'(4)} = -\frac{1}{2}$

4. Without solving for the inverse function $f^{-1}(x)$, find the derivative of the inverse function at the given x value:

(a) $f(x) = x^3 + 2x + 1, x = 1.$

$f'(x) = 3x^2 + 2.$ Also, since $f(0) = 1, f^{-1}(1) = 0,$ and $f'(0) = 2.$

Therefore, by Corollary 7.8, $D_x(f^{-1}(1)) = \frac{1}{f'(0)} = \frac{1}{2}.$

(b) $f(x) = \sqrt{x^3 + 2x + 1}, x = 2$

$f'(x) = \frac{1}{2}(x^3 + 2x + 1)^{-\frac{1}{2}}(3x^2 + 2) = \frac{3x^2 + 2}{2\sqrt{x^3 + 2x + 1}}.$

Also, since $f(1) = \sqrt{4} = 2, f^{-1}(2) = 1,$ and $f'(1) = \frac{5}{2\sqrt{4}} = \frac{5}{4}.$

Therefore, by Corollary 7.8, $D_x(f^{-1}(2)) = \frac{1}{f'(1)} = \frac{4}{5}.$

5. Find the derivative of each of the following functions:

(a) $f(x) = \ln(\sec x + \tan x)$

$f'(x) = \frac{1}{\sec x + \tan x} \cdot (\sec x \tan x + \sec^2 x) = \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x} = \sec x$

(b) $f(x) = \ln \sqrt{\frac{x^3}{x^5 + 1}}$

First, $f(x) = \frac{1}{2} [\ln(x^3) - \ln(x^5 + 1)] = \frac{3}{2} \ln x - \frac{1}{2} \ln(x^5 + 1)$

Therefore, $f'(x) = \frac{3}{2x} - \frac{5x^4}{2(x^5 + 1)}$

(c) $f(x) = (x^5 - 4x^3 - 7)^8(1 - x^2)^7(5x + 10)^{12}$

Using logarithmic differentiation, let $g(x) = \ln [(x^5 - 4x^3 - 7)^8(1 - x^2)^7(5x + 10)^{12}]$
 $= 8 \ln(x^5 - 4x^3 - 7) + 7 \ln(1 - x^2) + 12 \ln(5x + 10).$

Then $g'(x) = \frac{8(5x^4 - 12x^2)}{x^5 - 4x^3 - 7} - \frac{7(2x)}{1 - x^2} + \frac{12(5)}{5x + 10}.$

Hence $f'(x) = f(x)g'(x) = (x^5 - 4x^3 - 7)^8(1 - x^2)^7(5x + 10)^{12} \left[\frac{40x^4 - 96x^2}{x^5 - 4x^3 - 7} - \frac{14x}{1 - x^2} + \frac{60}{5x + 10} \right]$

(d) $f(x) = e^{3x^2} \sec^{-1}(2x^3)$

$f'(x) = 6xe^{3x^2} \sec^{-1}(2x^3) + e^{3x^2} \cdot \frac{6x^2}{2x^3\sqrt{4x^6 - 1}} = 6xe^{3x^2} \sec^{-1}(2x^3) + e^{3x^2} \cdot \frac{3}{x\sqrt{4x^6 - 1}}$

(e) $f(x) = e^{\ln(\cos(\sqrt{x}))}$

Notice that $f(x) = \cos(\sqrt{x}),$ so $f'(x) = -\sin(\sqrt{x}) \cdot \frac{1}{2}x^{-\frac{1}{2}} = -\frac{\sin \sqrt{x}}{2\sqrt{x}}.$

(f) $f(x) = x3^{-x} \log_5(7x^2)$

$f'(x) = 3^{-x} \log_5(7x^2) - x \ln(3)3^{-x} \log_5(7x^2) + x3^{-x} \frac{14x}{(\ln 5)(7x^2)}$

6. Find all extrema and inflection points of the function $f(x) = x^2e^{-x}$

$f'(x) = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2 - x),$ so $f(x)$ has critical numbers $x = 0$ and $x = 2.$

$f''(x) = 2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2e^{-x} = 2e^{-x} - 4xe^{-x} + x^2e^{-x}.$

If $f''(x) = 2e^{-x} - 4xe^{-x} + x^2e^{-x} = 0,$ then $2 - 4x + x^2 = 0.$

Applying the quadratic formula, this has solutions: $x = \frac{4 \pm \sqrt{16 - 4(1)(2)}}{2} = 2 \pm \sqrt{2}$

Therefore, f has extrema at $(0, 0)$ and $(2, \frac{4}{e^2})$ and inflection points at $(2 + \sqrt{2}, \frac{6 + 4\sqrt{2}}{e^{2 + \sqrt{2}}})$ and $(2 - \sqrt{2}, \frac{6 - 4\sqrt{2}}{e^{2 - \sqrt{2}}})$

7. Find an equation to the tangent line to the graph of $ye^{2x} + x \ln |y| = 1 + \tan^{-1}(x)$ at the point $(0, 1).$

Differentiating implicitly, $y'e^{2x} + y(2e^{2x}) + \ln |y| + x \cdot \frac{1}{y} \cdot y' = \frac{1}{1+x^2}$

Substituting $x = 0$ and $y = 1, y'(e^0) + 1(2e^0) + \ln(1) + (0) = \frac{1}{1},$ or $y' + 2 = 1.$ Thus $y' = -1.$

Then, by the point/slope formula: $y - 1 = -1(x - 0),$ or $y = -x + 1$ is the equation of the tangent line to the point $(0, 1).$

8. Find the area of the region bounded by the graphs of $y = xe^x$, $x = 0$, $x = 2$, and $y = 0$.

$$A = \int_0^2 xe^x dx = xe^x \Big|_0^2 - \int_0^2 e^x dx$$

$$u = x \quad dv = e^x dx$$

$$du = dx \quad v = e^x$$

$$= xe^x \Big|_0^2 - e^x \Big|_0^2 = (2e^2 - 0) - (e^2 - 1) = e^2 + 1.$$

9. Suppose that the population of a bacterial colony initially has 400 cells. After an hour, the population has increased to 800 cells. Find an equation for the population at any time. Then determine the population of the colony after 10 hours.

First recall that $f(x) = Ae^{kt}$ is the general formula for exponential population growth. Next, $f(0) = Ae^0 = A = 400$.

Also, $f(1) = 400e^k = 800$, so $2 = e^k$, and so $k = \ln 2$.

Therefore, there are $f(10) = 400e^{10 \ln(2)} = 409,600$ bacterial cells in the colony after 10 hours.

Or you could just notice that since the population doubles every hour, the population after 10 hour is 2^{10} times the original population.

10. Scientists trying to date the age of a fossil estimate that it contains 20% of the carbon-14 originally present. Given that the half-life of carbon-14 is 5730 years, approximately how old is the fossil?

First recall that $f(x) = Ae^{kt}$ is the general formula for exponential decay.

Also, since the half-life of Carbon-14 is 5730 years, $e^{5730k} = \frac{1}{2}$, so $.5 = e^{5730k}$, and so $k = \frac{\ln 2}{5730}$.

Therefore, to find the age of the fossil, we note that $.2A = Ae^{kt}$ or $.2 = e^{kt}$.

Thus $\ln .2 = kt$, or $\ln .2 = \frac{\ln 2}{5730}t$. Hence $t = \frac{5730 \ln .2}{\ln .5} \approx 13,305$ years.

11. According to Newton's Law of Cooling, the temperature of an object placed in a room with ambient temperature T_a satisfies the differential equation $y'(t) = k[y(t) - T_a]$.

- (a) Use separation of variables to find a general solution to this differential equation.

Let $y(t) = T$ and $y'(t) = \frac{dT}{dt}$. Then this differential equation can be written: $\frac{dT}{dt} = k(T - T_a)$.

Then, separating the variables, $\int \frac{1}{T - T_a} dT = \int k dt$.

That is, $\ln |T - T_a| = kt + C$, or $T - T_a = \pm e^{kt+C} = \pm e^{kt} \cdot e^C$.

If we let $A = \pm e^C$, then $y(t) = T = Ae^{kt} + T_a$ is the general solution to this differential equation.

- (b) Suppose a bowl of porridge initially at 200°F (too hot) is placed in a 70°F room. One minute later, the porridge has cooled down to 180°F . How long will it take for the porridge to cool down to 120°F (just right)?

Here, $T(0) = 200^\circ\text{F}$, and $T_a = 70^\circ\text{F}$. Then $y(0) = 200 = Ae^0 + 70$, so $A = 130$.

Next, $y(1) = 130e^k + 70 = 180$, so $130e^k = 110$, so $e^k = \frac{110}{130}$, so $k = \ln \frac{11}{13}$.

Therefore, $120 = 130e^{\ln \frac{11}{13} \cdot t} + 70$, so $\frac{50}{130} = e^{\ln \frac{11}{13} \cdot t}$.

Hence $\ln \frac{5}{13} = \ln \frac{11}{13} \cdot t$, so $t = \frac{\ln \frac{5}{13}}{\ln \frac{11}{13}} \approx 5.72$ minutes.

12. Compute each of the following *exactly*.

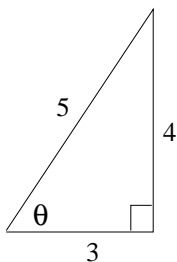
(a) $\arctan(-1) = -\frac{\pi}{4}$

(b) $\arcsin(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$

(c) $\arccos -\frac{\sqrt{2}}{2} = \frac{3\pi}{4}$

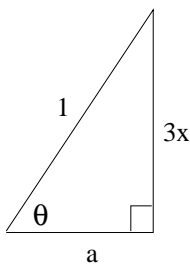
13. Simplify each of the following expressions:

(a) $\tan(\cos^{-1}(\frac{3}{5}))$



Using the Pythagorean Theorem, $5^2 = 3^2 + y^2$, or $25 - 9 = y^2$. Therefore, $y^2 = 16$ or $y = 2$.
Hence $\tan(\cos^{-1}(\frac{3}{5})) = \frac{4}{3}$.

(b) $\cot(\sin^{-1}(3x))$



Similarly, $(3x)^2 + a^2 = 1$, or $a^2 = 1 - 9x^2$. Hence $a = \sqrt{1 - 9x^2}$.
Therefore, $\cot(\sin^{-1}(3x)) = \frac{a}{3x} = \frac{\sqrt{1-9x^2}}{3x}$.

14. Find all solutions to the equation $10 \cos^2 x - \cos x = 3$ on the interval $[0, 2\pi)$

Notice that this equation has quadratic form. Therefore, if we substitute $u = \cos x$, we have $10u^2 - u = 3$.

Rearranging this, $10u^2 - u - 3 = 0$, or, factoring, $(2u + 1)(5u - 3) = 0$. This either $2u + 1 = 0$ or $5u - 3 = 0$

Then either $\cos x = -\frac{1}{2}$, in which case $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$,

or $\cos x = \frac{3}{5}$, in which case, $x \approx .9273$ or $x \approx 2\pi - .9273 \approx 5.3559$

15. Evaluate the following integrals:

(a) $\int \frac{3x^2}{1+x^6} dx$

Let $u = x^3$. Then $du = 3x^2 dx$, and $\int \frac{3x^2}{1+x^6} dx = \int \frac{1}{1+u^2} du = \arctan u + C = \arctan(x^3) + C$

(b) $\int \frac{4x}{\sqrt{1-x^4}} dx$

Let $u = x^2$. Then $du = 2x dx$ and $\int \frac{4x}{\sqrt{1-x^4}} dx = 2 \int \frac{1}{\sqrt{1-u^2}} du = 2 \arcsin(u) + C = 2 \arcsin(x^2) + C$

(c) $\int_{\sqrt{2}}^2 \frac{4}{x\sqrt{x^2-1}} dx$

$= 4 \sec^{-1}(x) \Big|_{\sqrt{2}}^2 = 4[\sec^2(x) - \sec^{-1}(\sqrt{2})] = 4[\cos^{-1}(\frac{1}{2}) - \cos^{-1}(\frac{\sqrt{2}}{2})] = 4(\frac{\pi}{3} - \frac{\pi}{4}) = 4(\frac{\pi}{12}) = \frac{\pi}{3}$

(d) $\int_3^4 x\sqrt{x-3} dx$

Let $u = x - 3$. Then $du = dx$, and $u + 3 = x$. Therefore, $\int_3^4 x\sqrt{x-3} dx = \int_0^1 (u+3)u^{\frac{1}{2}} du$

$= \int_0^1 u^{\frac{3}{2}} + 3u^{\frac{1}{2}} du = \frac{2}{5}u^{\frac{5}{2}} + (3)\frac{2}{3}u^{\frac{3}{2}} \Big|_0^1 = \left[\left(\frac{2}{5} + 2\right) - (0) \right] = \frac{12}{5}$

(e) $\int_1^e x^2 \ln x \, dx$

Let $u = \ln x$ and $dv = x^2 \, dx$

Then $du = \frac{1}{x} \, dx$ and $v = \frac{1}{3}x^3$

Thus, using integration by parts: $\int_1^e x^2 \ln x \, dx = \frac{1}{3}x^3 \ln x \Big|_1^e - \frac{1}{3} \int_1^e x^2 \, dx$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 \Big|_1^e = \left[\frac{1}{3}e^3 - \frac{1}{9}e^3 \right] - \left[0 - \frac{1}{9} \right] = \frac{2}{9}e^3 + \frac{1}{9}$$

(f) $\int x^2 e^{-3x} \, dx$

Let $u = x^2$ and $dv = e^{-3x} \, dx$

Then $du = 2x \, dx$ and $v = -\frac{1}{3}e^{-3x}$

Thus, using integration by parts: $\int x^2 e^{-3x} \, dx = -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3} \int x e^{-3x} \, dx$.

Now let $u = x$ and $dv = e^{-3x} \, dx$

Then $du = dx$ and $v = -\frac{1}{3}e^{-3x}$

Thus, using integration by parts: $\int x^2 e^{-3x} \, dx = -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3} \left[-\frac{1}{3}x e^{-3x} + \int e^{-3x} \, dx \right]$

$$= -\frac{1}{3}x^2 e^{-3x} - \frac{2}{9}x e^{-3x} - \frac{2}{27}e^{-3x} + C.$$

(g) $\int e^{5x} \sin x \, dx$

Let $u = e^{5x}$ and $dv = \sin x \, dx$

Then $du = 5e^{5x} \, dx$ and $v = -\cos x$

Thus, using integration by parts: $\int e^{5x} \sin x \, dx = -\cos x e^{5x} + 5 \int e^{5x} \cos x \, dx$.

Let $u = e^{5x}$ and $dv = \cos x \, dx$

Then $du = 5e^{5x} \, dx$ and $v = \sin x$

Thus, using integration by parts: $\int e^{5x} \sin x \, dx = -\cos x e^{5x} + 5 \left[\sin x e^{5x} - 5 \int e^{5x} \sin x \, dx \right]$

$$= -\cos x e^{5x} + 5 \sin x e^{5x} - 25 \int e^{5x} \sin x \, dx.$$

Therefore, $26 \int e^{5x} \sin x \, dx = -\cos x e^{5x} + 5 \sin x e^{5x}$.

$$\text{Hence } \int e^{5x} \sin x \, dx = -\frac{1}{26} \cos x e^{5x} + \frac{5}{26} \sin x e^{5x}.$$

(h) $\int x^3 e^{x^2} \, dx$

Let $w = x^2$. Then $dw = 2x \, dx$, or $\frac{1}{2}dw = x \, dx$

So $\int x^3 e^{x^2} \, dx = \frac{1}{2} \int w e^w \, dw$.

Next, let $u = w$ and $dv = e^w \, dw$

Then $du = dw$ and $v = e^w$

Therefore, $\int x^3 e^{x^2} \, dx = \frac{1}{2} \int w e^w \, dw = \frac{1}{2} \left[w e^w - \int e^w \, dw \right]$.

$$\text{Thus } \int x^3 e^{x^2} \, dx = \frac{1}{2} [w e^w - e^w] + C.$$

$$(i) \int \cos^{-1} x \, dx$$

Let $u = \cos^{-1} x$ and $dv = dx$

Then $du = -\frac{1}{\sqrt{1-x^2}} dx$ and $v = x$

Thus, using integration by parts: $\int \cos^{-1} x \, dx = x \cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}} dx$

Now, let $w = 1 - x^2$. Then $dw = -2x dx$, so $-\frac{1}{2}dw = x dx$.

$$\begin{aligned} \text{Therefore, } \int \cos^{-1} x \, dx &= x \cos^{-1} x - \frac{1}{2} \int \frac{1}{\sqrt{w}} dw \\ &= x \cos^{-1} x - \frac{1}{2} \int w^{-\frac{1}{2}} dw = x \cos^{-1} x - \frac{1}{2} [2w^{\frac{1}{2}}] + C \\ &= x \cos^{-1} x - \sqrt{1-x^2} + C \end{aligned}$$

$$(j) \int \sin^3 x \cos^3 x \, dx$$

Recall that $\cos^2 x = 1 - \sin^2 x$. Therefore, $\int \sin^3 x \cos^3 x \, dx = \int \sin^3 x (1 - \sin^2 x) \cos x \, dx = \int (\sin^3 x - \sin^5 x) \cos x \, dx$.

Let $u = \sin x$. Then $du = \cos x \, dx$, and $\int (\sin^3 x - \sin^5 x) \cos x \, dx = \int (u^3 - u^5) du$.

$$= \frac{1}{4}u^4 - \frac{1}{6}u^6 + C = \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$$

$$(k) \int \sin^2 x \cos^2 x \, dx$$

Using the half angle reduction formulas: $\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos(2x)$ and $\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos(2x)$,

$$\begin{aligned} \int \sin^2 x \cos^2 x \, dx &= \int \left(\frac{1}{2} - \frac{1}{2} \cos(2x) \right) \left(\frac{1}{2} + \frac{1}{2} \cos(2x) \right) dx = \int \frac{1}{4} - \frac{1}{4} \cos^2(2x) \, dx = \int \frac{1}{4} - \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \cos(4x) \right) dx \\ &= \int \frac{1}{4} - \frac{1}{8} - \frac{1}{8} \cos(4x) \, dx = \int \frac{1}{8} - \frac{1}{8} \cos(4x) \, dx = \frac{1}{8}x - \frac{1}{32} \sin(4x) + C \end{aligned}$$

$$(l) \int \tan^3 x \sec^3 x \, dx$$

Recall the Pythagorean identity; $\tan^2 x + 1 = \sec^2 x$ or $\tan^2 x = \sec^2 x - 1$

Then $\int \tan^3 x \sec^3 x \, dx = \int \tan x \tan^2 x \sec^2 x \sec x \, dx = \int \tan x \tan^2 x \sec^2 x \sec x \tan x \, dx$

$$= \int (\sec^2 x - 1) \sec^2 x \sec x \tan x \, dx.$$

Let $u = \sec x$. Then $du = \sec x \tan x \, dx$, and we have:

$$\begin{aligned} \int (u^2 - 1) u^2 \, du &= \int u^4 - u^2 \, du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \\ &= \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C \end{aligned}$$

$$(m) \int \tan^6 x \, dx$$

We will once again use the identity $\tan^2 x = \sec^2 x - 1$.

Then $\int \tan^6 x \, dx = \int (\sec^2 x - 1) (\sec^2 x - 1) (\sec^2 x - 1) \, dx$

$$= \int (\sec^4 x - 2\sec^2 x + 1) (\sec^2 x - 1) \, dx = \int \sec^6 x - 3\sec^4 x + 3\sec^2 x - 1, \, dx$$

$$= \int \sec^2 x \cdot \sec^2 x \cdot \sec^2 x \, dx - 3 \int \sec^2 x \sec^2 x \, dx + 3 \int \sec^2 x - 1 \, dx$$

$$= \int (\tan^2 x + 1) \cdot (\tan^2 x + 1) \cdot \sec^2 x \, dx - 3 \int (\tan^2 x + 1) \cdot \sec^2 x \, dx + \int 3 \sec^2 x - 1 \, dx$$

Let $u = \tan x$. Then $du = \sec^2 x \, dx$ and we have:

$$= \int (u^2 + 1)(u^2 + 1) \, du - 3 \int u^2 + 1 \, du + \int 3 \sec^2 x - 1 \, dx$$

$$= \int u^4 + 2u^2 + 1 \, du + \int -3u^2 - 3 \, du + \int 3 \sec^2 x - 1 \, dx$$

$$= \frac{1}{5}u^5 + \frac{2}{3}u^3 + u - u^3 - 3u + 3 \tan x - x + C = \frac{1}{5}u^5 - \frac{1}{3}u^3 - 2u + 3 \tan x - x + C$$

$$= \frac{1}{5} \tan^5 x - \frac{1}{3} \tan^3 x - 2 \tan x + 3 \tan x - x + C = \frac{1}{5} \tan^5 x - \frac{1}{3} \tan^3 x + \tan x - x + C$$

(n) $\int (\tan x + \cot x)^2 \, dx$

Multiplying out, $\int (\tan x + \cot x)^2 \, dx = \int \tan^2 x + 2 \tan x \cot x + \cot^2 x \, dx = \int \tan^2 x + 2 + \cot^2 x \, dx$

Using the identities: $\tan^2 x = \sec^2 x - 1$ and $\cot^2 x = \csc^2 x - 1$, we have:

$$= \int \sec^2 x - 1 + 2 + \csc^2 x - 1 \, dx = \int \sec^2 x + \csc^2 x \, dx$$

$$= \tan x - \cot x + C$$

16. Evaluate the following integrals:

(a) $\int \frac{1}{1+x^2} dx = \arctan(x) + C$

(b) $\int \frac{x}{1+x^2} dx$

Let $u = 1 + x^2$. Then $du = 2x dx$, or $\frac{1}{2}du = x dx$.

Therefore, $\int \frac{x}{1+x^2} dx = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln|x^2 + 1| + C = \frac{1}{2} \ln(x^2 + 1) + C$

(c) $\int \frac{x^2}{1+x^2} dx = \int \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2} dx = \int 1 - \frac{1}{1+x^2} dx$
 $= x - \arctan(x) + C$

(d) $\int \frac{x^3}{1+x^2} dx$

Let $u = x^2 + 1$. Then $du = 2x dx$ or $\frac{1}{2}du = x dx$ and $u - 1 = x^2$.

Therefore, $\int \frac{x^3}{1+x^2} dx = \int \frac{x^2 \cdot x}{1+x^2} dx = \frac{1}{2} \int \frac{(u-1)}{u} du = \frac{1}{2} \int 1 - \frac{1}{u} du$
 $= \frac{1}{2}[u - \ln|u|] + C = \frac{x^2+1}{2} - \frac{1}{2} \ln(x^2 + 1) + C$

(e) $\int \frac{x^4}{1+x^2} dx = \int x \cdot \frac{x^3}{1+x^2} dx$

Note: There are many ways to work this problem, and what follows is not the easiest way, but I am choosing this method to show you how so solve it using integration by parts. Let me know if you find one of the simpler methods.

Let $u = x$ and $dv = \frac{x^3}{1+x^2}$.

Then $du = dx$, and, from above, $v = \frac{x^2+1}{2} - \frac{1}{2} \ln(x^2 + 1) = \frac{x^2}{2} + \frac{1}{2} - \frac{1}{2} \ln(x^2 + 1)$

Therefore, using integration by parts, $\int x \cdot \frac{x^3}{1+x^2} dx = x \left(\frac{x^2+1}{2} - \frac{1}{2} \ln(x^2 + 1) \right) - \int \left(\frac{x^2}{2} + \frac{1}{2} - \frac{1}{2} \ln(x^2 + 1) \right) dx$
 $= \frac{x^3}{2} + \frac{x}{2} - \frac{x \ln(x^2 + 1)}{2} - \frac{x^3}{6} - \frac{x}{2} + \frac{1}{2} \int \ln(x^2 + 1) dx$

Now, to integrate $\frac{1}{2} \int \ln(x^2 + 1) dx$:

Let $u = \ln(x^2 + 1)$ and $dv = dx$ Then $du = \frac{2x}{x^2+1}$ and $v = x$.

Therefore, $\frac{1}{2}[x \ln(x^2 + 1)] - \frac{1}{2} \int \frac{2x^2}{x^2 + 1} dx = \frac{x \ln(x^2 + 1)}{2} - \int \frac{x^2}{x^2 + 1} dx$

Which, from above, is $\frac{x \ln(x^2 + 1)}{2} - x + \arctan(x) + C$

Hence $\int \frac{x^4}{1+x^2} dx = \frac{x^3}{2} + \frac{x}{2} - \frac{x \ln(x^2 + 1)}{2} - \frac{x^3}{6} - \frac{x}{2} + x \ln(x^2 + 1) - x + \arctan(x) + C = \frac{x^3}{3} - x + \arctan(x) + C$