Math 262
Mass, Moments, and Center of Mass

1. **Mass:**

   (a) Suppose that L is 2-dimensional lamina in the shape of a region R in the plane that is bounded above by $f(x)$ and below by $g(x)$ on $[a,b]$. Further suppose that $\rho(x)$ is a function that gives the density per unit area at each point of the lamina [so the lamina has constant density along vertical slices of the lamina]. Then the mass of L is given by:

   $$m = \int_a^b [f(x) - g(x)]\,\rho(x)\,dx$$

   (b) Now suppose that L is 2-dimensional lamina in the shape of a region R in the plane that is bounded on the right by $f(y)$ and on the left by $g(y)$ on $[c,d]$. Further suppose that $\rho(y)$ is a function that gives the density per unit area at each point of the lamina [so the lamina has constant density along horizontal slices of the lamina]. Then the mass of L is given by:

   $$m = \int_c^d [f(y) - g(y)]\,\rho(y)\,dy$$

2. **First Moments and the Center of Mass:**

   (a) We again suppose that L is 2-dimensional lamina in the shape of a region R in the plane that is bounded above by $f(x)$ and below by $g(x)$ on $[a,b]$. Further suppose that $\rho(x) = k$ is a function that gives the density per unit area at each point of the lamina [so the lamina has uniform (constant) density]. Then the first moment of L with respect to the x and y axes are:

   $$M_x = \int_a^b \frac{1}{2}\,[f(x) + g(x)]\,[f(x) - g(x)]\,\rho(x)\,dx$$

   and

   $$M_y = \int_a^b x \cdot [f(x) - g(x)]\,\rho(x)\,dx$$

   The center of mass of the lamina L is:
   $(\overline{x}, \overline{y})$, where $\overline{x} = \dfrac{M_y}{m}$ and $\overline{y} = \dfrac{M_x}{m}$

   (b) Now suppose that L is 2-dimensional lamina in the shape of a region R in the plane that is bounded on the right by $f(y)$ and on the left by $g(y)$ on $[c,d]$. Further suppose that $\rho(y)$ is a function that gives the density per unit area at each point of the lamina [so the lamina has constant density along horizontal slices of the lamina]. Then the first moment of L with respect to the x and y axes are:

   $$M_x = \int_c^d y \cdot [f(y) - g(y)]\,\rho(y)\,dy$$

   and

   $$M_y = \int_c^d \frac{1}{2}\,[f(y) + g(y)]\,[f(y) - g(y)]\,\rho(y)\,dy$$

   The center of mass of the lamina L is:
   $(\overline{x}, \overline{y})$, where $\overline{x} = \dfrac{M_y}{m}$ and $\overline{y} = \dfrac{M_x}{m}$