Introduction to Series

Definitions:

• An **infinite series** (or simply a **series**) is an expression of the form: $a_1 + a_2 + ... + a_n + ...$
or, in summation notation: $\sum_{n=1}^{\infty} a_n$, or, more simply (but vaguely) written: $\sum a_n$

• The k**th partial sum**, S_k, of a series $\sum_{n=1}^{\infty} a_n$ is:

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + ... + a_k$$

• The **sequence of partial sums** of the series $\sum a_n$ is:

$$\{S_n\} = \{S_1, S_2, ..., S_n, ...\}$$

• A series $\sum a_n$ is **convergent** if its associated sequence of partial sums $\{S_n\}$ converges. That is, if $\lim_{n\to\infty} S_n = S$ for some real number S.

• This real number S is called the **sum** of the series $\sum a_n$, and we write $S = a_1 + a_2 + ... + a_n + ...$

• A series $\sum a_n$ is **divergent** if its associated sequence of partial sums $\{S_n\}$ diverges. In this case, we say that the series has no sum.

Theorem 11.14: The **harmonic series** is the series $1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} + ...$

Claim: The harmonic series diverges. (Proof: next time.)

Important Examples: A **geometric series** is any series of the form:
$a + ar + ar^2 + ... + +ar^n + ...$, where a and r are real numbers and $a \neq 0$.

Theorem 11.15: Let $a \neq 0$. The geometric series $a + ar + ar^2 + ... + ar^n + ...$
(I) Converges and has the sum $S = \frac{a}{1-r}$ if $|r| < 1$.

(II) Diverges if $|r| \geq 1$.

Proof: (next time)

Theorem 11.16: If a series $\sum a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

Proof: Notice that $S_n - S_{n-1} = a_n$. Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - S_{n-1}$.
Since $\sum a_n$ is convergent, $\lim_{n\to\infty} S_n = S$ for some real number S. But then $\lim_{n\to\infty} S_{n-1} = S$ as well.
Therefore, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - S_{n-1} = S - S = 0$.