Series Intro Part 2

Recall:

- The **harmonic series**, defined as $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
- According to Theorem 11.16, If a series $\sum a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

Combining these two pieces of information, we have the following result:

The nth Term Test Given the series $\sum a_n$:
(I) If $\lim_{\to\infty} a_n \neq 0$, the the series $\sum a_n$ is divergent.
(II) If $\lim_{\to\infty} a_n = 0$, further investigation is needed, as the series might converge, and also might diverge.

Theorem 11.8 If $\sum a_n$ and $\sum b_n$ are series such that $a_j = b_j$ for every $j > k$, where k is a positive integer, then both series converge or both series diverge.

Proof: Since $a_n = b_n$ for $j > k$, then $\sum_{n=k+1}^{\infty} a_n = \sum_{n=k+1}^{\infty} b_n$.
Let $\sum_{n=1}^{k} a_n = N$ and $\sum_{n=1}^{k} b_n = M$.
Then, $\sum_{n=1}^{\infty} a_n = N + \sum_{n=k+1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n = M + \sum_{n=k+1}^{\infty} b_n$.
Hence either both of these converge, or both diverge.

Theorem 11.19 For any positive integer k, the series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ...$ and $\sum_{n=k+1}^{\infty} a_n = a_{k+1} + a_{k+2} + ...$ either both converge or both diverge.

Proof: Use the same basic argument as above (think of $\sum b_n$ where $b_n = 0$ for $n \leq k$ and $b_n = a_n$ for $n > k$).

Theorem 11.20 If $\sum a_n$ and $\sum b_n$ are both convergent series with sums A and B respectively, then:

(I) $\sum a_n + b_n = A + B$

(II) $\sum ca_n$ converges and has sum cA for every real number c.

(III) $\sum a_n - b_n = A - B$

Theorem 11.21: If $\sum a_n$ is a convergent series and $\sum b_n$ is a divergent series, then $\sum a_n + b_n$ is divergent.

Proof: Suppose that there was an example of a convergent series $\sum a_n$ and a divergent series $\sum b_n$ for which the series $\sum a_n + b_n$ was convergent.
Suppose that $\sum a_n = A$ and $\sum a_n + b_n = M$.
Then, by Theorem 11.20(III), $\sum[(a_n + b_n) - a_n] = \sum[a_n + b_n] - \sum a_n = M - A$, so this series converges.
But $\sum[(a_n + b_n) - a_n] = \sum b_n$, which diverges by hypothesis.
Since this is clearly preposterous, our assumption that there is an example of a convergent series $\sum a_n$ and a divergent series $\sum b_n$ for which the series $\sum a_n + b_n$ was convergent must be false.
This proves the theorem. [This method of proof is called an "indirect proof", or a "proof by contradiction"]