Series Tests Summary

1. The nth Term Test.

   Application: Can be used on *any* series $\sum_{n=1}^{\infty} a_n$.

   Conclusion: Allows us to conclude that the series **diverges** if $\lim_{n\to\infty} a_n \neq 0$.

   Other Comments: If $\lim_{n\to\infty} a_n = 0$, the **NO** conclusion can be reached!

2. The Geometric Series Test.

   Application: Can **only** be used on series of the form $\sum ar^n$.

   Conclusion: Allows us to conclude that the series **converges** to $S = \frac{a}{1-r}$ if $|r| < 1$. The series **diverges** if $|r| \geq 1$.

   Other Comments: Geometric series are one of the few series that we can find the sun of exactly. They are also useful in comparison tests for series whose form is similar to a geometric series.

3. The p-series Test.

   Application: Can **only** be used on series of the form $\sum \frac{1}{n^p}$.

   Conclusion: Allows us to conclude that the series **converges** if $p > 1$ and that he series **diverges** if $p \leq 1$.

   Other Comments: p-series are also useful in comparison tests for series whose form is similar to a p-series.

4. The Integral Test.

   Application: Can **only** be used on a positive term series given by a function $f(n)$ whose real valued analog $f(x)$ is **positive**, **continuous**, and **decreasing** on $[1, \infty)$, and these conditions *must* be verified.

   Conclusion: Allows us to conclude that the series **converges** if its related improper integral $\int_1^{\infty} f(x)\,dx$ converges and **diverges** if its related improper integral $\int_1^{\infty} f(x)\,dx$ diverges.

   Other Comments: This test is useful in certain circumstances but is often not the easiest test to use.

5. The Comparison Test.

   Application: Can **only** be used on a positive term series $\sum a_n$ and a positive term comparison series $\sum b_n$ whose convergence or divergence is known and which satisfies the necessary inequality.

   Conclusion: Allows us to conclude that the series $\sum a_n$ **converges** if $a_n \leq b_n$ for all n and $\sum b_n$ is known to converge. Similarly, $\sum a_n$ **diverges** if $a_n \geq b_n$ and $\sum b_n$ is known to diverge.

   Other Comments: This test only useful if we can *prove* the necessary inequality comparing the given series with one whose behavior is well understood.

6. The Limit Comparison Test.

   Application: Can **only** be used on a positive term series $\sum a_n$ and a positive term comparison series $\sum b_n$ whose convergence or divergence is known.

   Conclusion: Allows us to conclude that the series $\sum a_n$ **converges** if $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ and $\sum b_n$ is known to converge. Similarly, $\sum a_n$ **diverges** if $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ and $\sum b_n$ is known to diverge.

   Other Comments: This test only useful if we can compute the limit of the ratio of the two series and obtain a finite value. Especially useful for series that are similar to geometric or p-series.

7. The Ratio Test.

   Application: Can be used on a positive term series $\sum a_n$ to show convergence, or on $\sum |a_n|$ to show absolute convergence of an alternating series.

   Conclusion: Allows us to conclude that the series $\sum a_n$ **converges**(absolutely) if $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, **diverges** if $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ and is **inconclusive** if $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L = 1$

   Other Comments: This test is useful for series involving factorials or nth powers.

8. The Root Test.

   Application: Can be used on a positive term series $\sum a_n$ to show convergence, or on $\sum |a_n|$ to show absolute convergence of an alternating series.

   Conclusion: Allows us to conclude that the series $\sum a_n$ **converges**(absolutely) if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, **diverges** if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ and is **inconclusive** if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$

   Other Comments: This test is useful for series involving nth powers.

9. The Alternating Series Test.

   Application: Can be used on an alternating series $\sum (-1)^n a_n$ to show *conditional* convergence.

   Conclusion: Allows us to conclude that the series $\sum (-1^n) a_n$ **converges**(conditionally) if $a_k \geq a_{k+1}$ for every k and $\lim\limits_{n\to\infty} a_n = 0$.

   Other Comments: We must prove that the terms of the alternating series are non-increasing. Also, this test does not help us show that a series is divergent.