Math 310
Exam 2 - Practice Problems

1. Prove that there is an integer m such that $m^3 > 10^{10}$. Is your proof constructive or non-constructive?
2. Prove that given any two rational numbers $p < q$, there is a rational number r with $p < r < q$.
3. Prove that given a non-negative integer n, there is a unique non-negative integer m such that $m^2 \leq n < (m+1)^2$
4. Prove or disprove: Every non-negative integer can be written as the sum of at most 3 perfect squares.
5. Formulate a conjecture about the final two digits of the square of any integer. Then prove your conjecture using a proof by cases.
6. For each of the following, determine whether the statement is True or False.
   (a) $\emptyset \subseteq \{a, b, c, d\}$
   (b) $\emptyset \in \{a, b, c, d\}$
   (c) $\emptyset \in \{a, b, \emptyset\}$
   (d) $\emptyset \subseteq \{a, b, \emptyset\}$
   (e) $\{a, b\} \subset \{a, b\}$
   (f) $0 \in \{0, \{1\}, \{0, 1\}\}$
   (g) $1 \in \{0, \{1\}, \{0, 1\}\}$
   (h) $\{0, 1\} \in \{0, \{1\}, \{0, 1\}\}$
   (i) $\{0, 1\} \subset \{0, \{1\}, \{0, 1\}\}$
7. Given the set $B = \{a, b, \{a, b\}\}$
   (a) Find $|B|$.
   (b) Find $\mathcal{P}(B)$
8. Given that $A = \{1, 2, 3\}$ and $B = \{a, b, c, d, e, f\}$
   (a) List the elements in $A \times A$.
   (b) How many elements are in $A \times B$?
   (c) How many elements are in $A \times (B \times B)$?
9. Find the set of all elements that make the predicate $Q(x) : x^2 < x$ true (where the domain of x is all real numbers).
10. Given that $A = \{0, 2, 4, 6, 8, 10, 12\}$, $B = \{0, 2, 3, 5, 7, 11, 12\}$ and $C = \{1, 2, 3, 4, 6, 7, 8, 9\}$ are all subsets of the universal set $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, find each of the following:
    (a) $A - B$
    (b) $\overline{A}$
    (c) $A \cap B$
    (d) $A \cup (B \cap C)$
    (e) $A - (\overline{B} \oplus C)$
    (f) $(A \cap C) \cup (B - \overline{A})$
11. Draw Venn Diagrams representing each of the following sets:
    (a) $A - B$
    (b) $B - \overline{A}$
    (c) $(A \cup C) \cap B$
    (d) $\overline{A \cup B \cup C}$
    (e) $A - (B \cup C)$
    (f) $(A \cap B) - \overline{C}$
12. Use a membership table to show that $(B - A) \cup (C - A) = (B \cup C) - A$.
13. Use a 2-column proof to verify the set identity: $A \cup (A \cap B) = A$.
14. Use a paragraph (double containment) proof to show that $A - B = A \cap \overline{B}$.
15. For each of the following, either prove the statement or show that it is false using a counterexample.
    (a) $(A - B) - C = A - (B - C)$
    (b) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$
    (c) $A \cap (B - C) = (A \cap B) - (A \cap C)$
16. Prove that $n^5 - n$ is divisible by 5 for any non-negative integer n.

17. Prove that for $r \in \mathbb{R}$, $r \neq 1$ and for all integers n, $\displaystyle\sum_{j=0}^{n} r^j = \frac{r^{n+1}-1}{r-1}$

18. Prove that for all $n \geq 2$, $\displaystyle\sum_{k=1}^{n} \frac{1}{k^2} < 2 - \frac{1}{n}$

19. Prove that $n! < n^n$ whenever $n > 1$.

20. Prove that for all n, $\displaystyle\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$

21. Suppose that $f(x) = e^x$ and $g(x) = xe^x$. Use induction and the product rule to show that $g^{(n)}(x) = (x+n)e^x$ for all $n \geq 1$.

22. Given the relation $R = \{(1,1),(1,3),(1,4),(2,2),(3,1),(3,4),(4,1),(4,3)\}$ on the set $A = \{1,2,3,4\}$:

    (a) Determine whether or not R is reflexive, irreflexive, symmetric, antisymmetric, transitive.
    (b) Find the matrix representation M_R for this relation.
    (c) Draw the graph representation of this relation Γ_R.

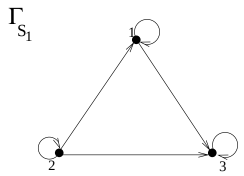
23. Given the relation $S = \{(1,1),(1,3),(2,1),(2,2),(2,3),(3,3),(4,4)\}$ on the set $A = \{1,2,3,4\}$:

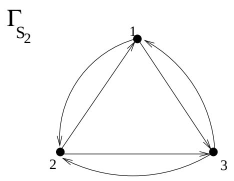
    (a) Determine whether or not S is reflexive, irreflexive, symmetric, antisymmetric, transitive.
    (b) Find the matrix representation M_S for this relation.
    (c) Draw the graph representation of this relation Γ_S.

24. Let R_1 and R_2 be given by the matrices $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

    (a) Determine whether or not R_1 is reflexive, irreflexive, symmetric, antisymmetric, transitive.
    (b) Determine whether or not R_2 is reflexive, irreflexive, symmetric, antisymmetric, transitive.
    (c) Find the matrix representing $\overline{R_1}$, $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 \oplus R_2$, and $R_1 \circ R_2$
    (d) Draw the graphs representing R_1 and R_2.

25. Given the graphs representing the relations S_1 and S_2:





    (a) Determine whether or not R_1 is reflexive, irreflexive, symmetric, antisymmetric, transitive.
    (b) Determine whether or not R_2 is reflexive, irreflexive, symmetric, antisymmetric, transitive.
    (c) Draw the graph representing $\overline{S_2}$, $S_1 \cup S_2$, $S_1 \cap S_2$, $S_2 - S_1$, and $S_2 \circ S_1$
    (d) Find the matrix representing S_1.
    (e) List the ordered pairs in S_2.

26. Suppose that R and S are symmetric relations on a non-empty set A. Prove or disprove each of these statements:

    (a) $R \cup S$ is symmetric.
    (b) $R \cap S$ is symmetric.
    (c) $R - S$ is symmetric.
    (d) $R \oplus S$ is symmetric.
    (e) $S \circ R$ is symmetric.

27. Prove or disprove: given two relations R and S on a set A, $R \circ S = S \circ R$