Math 102
Exam 2 - Version 1: Solutions **Name:**________________________________
10/08/2010

1. (3 points each) Compute the value of each of the following:

   (a) 7!

   $= 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5,040$

   (c) $C(9, 4)$

   $C(9,4) = \frac{9!}{5!4!} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{24} = 126$

   (b) $\dfrac{59!}{57!}$

   $= 59 \cdot 58 = 3,422$

   (d) $P(6, 4)$

   $P(6,4) = \frac{6!}{2!} = \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 360$

2. (5 points) Suppose you need to choose an identification code consisting of two numbers followed by 3 letters. How many possible codes can be formed if you can repeat any number and any letter as many times as you want. (For example: 55BBB would be allowed).

   Since there are 10 possible digits, 26 possible letters, and repetition is allowed, there are:

   $10 \cdot 10 \cdot 26 \cdot 26 \cdot 26 = 1,757,600$ different identification codes possible.

3. (5 points) Suppose you need to choose an identification code consisting of two numbers followed by 3 letters. How many possible codes can be formed if you *can* repeat any number but you *cannot* repeat a letter (For example: 33BBB and 47BCC would both *not* be allowed, but 44QAW would be allowed).

   Since repetition of letters is not allowed, we adjust our slot diagram from above by removing an option each time a letter is chosen. Then there are:

   $10 \cdot 10 \cdot 26 \cdot 25 \cdot 24 = 1,560,000$ different identification codes possible.

4. (5 points) Suppose you need to choose an identification code consisting of two numbers followed by 3 letters. How many possible codes can be formed if you *cannot* repeat any letters, and the sum of the numbers used must add up to 5 (For example: 23AFY would be allowed, but 23AYA and 57ZXD would both *not* be allowed).

   Notice that since the only codes allowed are those in which the sum of the numbers is 5, then the code must begin with either: 50, 41, 32, 23, 14, or 05.
   Then there are 6 different choices for the numerical portion of the code. The letters are chosen as above.

   Therefore, there are $(6 \cdot 26 \cdot 25 \cdot 24) = 93,600$ different identification codes possible.

5. Suppose that the Parliament of a certain country consists of 10 monarchists and 8 anarchists.

   (a) (5 points) How many ways can a ruling council of 5 people be chosen from among all the members of Parliament?

   Notice that since a council is being chosen, order does not matter, so this is combination counting.
   Then we have $C(18,5) = \frac{18!}{5!13!} = \frac{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14}{5!} = 8,568$
   So there are 8,568 different ways to choose a ruling council from among the 18 members of Parliament.

   (b) (5 points) How many ways can a ruling council of 5 people be chosen from among all the members of Parliament if 3 of them must be monarchists and 2 of them must be anarchists?

   Here, since we want a certain number of monarchists and certain number of anarchists, this is a "double combination" problem
   We have $C(10,3) \cdot = \frac{10!}{7!3!} = \frac{10 \cdot 9 \cdot 8}{3!} = 120$ ways to choose the three monarchists.
   We have $C(8,2) = \frac{8!}{6!2!} = \frac{8 \cdot 7}{2} = 28$ ways to choose the two anarchists.
   Then there are $120 \cdot 28 = 3,360$ different ways to choose the members of the ruling council under these conditions.

6. (6 points each) Suppose that Andrew, Brian, Charles, Doug, Amy, Betty, Carina, and Darlene are all members of the Machiavellian Club on campus. They need to elect their club officers: President, Vice President, and Treasurer.

   (a) How many different ways could these offices be filled?

   Since there are different offices, the order that we assign people of offices matters. We assume that a club member cannot hold two offices, so we use either permutation counting or a slot diagram. Notice that there are 8 total club members. We will count by choosing the President first, then the VP, then the Treasurer.
   Then there are $8 \cdot 7 \cdot 6 = 336$ different ways that the office could be filled under these assumptions.

   (b) Suppose that Charles will only agree to serve as an officer if he gets to be President. How many ways could the offices be filled under these circumstances?

   The key thing to notice here is that just because Charles refuses to hold any office besides President **does not** mean that he will end up being elected president. It only means that he will refuse both other offices. To count this, we will need to consider two cases:
   First, if Charles is elected president, then we count the ways of filling the offices as follows (there are still two remaining offices to fill):
   $1 \cdot 7 \cdot 6 = 42$ ways to fill the other offices if Charles is elected president.
   Next, if Charles is not elected president, then he is completely removed from consideration (since he will not take another office), so we can think of this as filling the offices by choosing from among the other 6 club members:
   $7 \cdot 6 \cdot 5 = 210$.
   Since these are two separate cases involving different initial assumptions, we find the total number of ways that the offices could be filled by adding the totals from these two cases:
   There are $42 + 210 = 252$ ways of filling the three offices under these conditions.

   (c) Now suppose that Charles will only agree to serve as an officer if he gets to be President. Also, suppose that Darlene hates Charles and refuses to be an officer if he gets elected to an office. How many ways could the offices be filled under these circumstances?

   Notice that this situation is similar to the previous one, with the exception that our first cases changes. Now, if Charles is elected President of the club, Darlene will refuse to serve as as officer, so there will be one fewer choice available in that case. To count this, we will again need to consider two cases:
   First, if Charles is elected president, then we count the ways of filling the offices as follows (there are still two remaining offices to fill):
   $1 \cdot 6 \cdot 5 = 30$ ways to fill the other offices if Charles is elected president (Darlene is no longer an option here).
   Next, if Charles is not elected president, then he is completely removed from consideration (since he will not take another office), so we can think of this as filling the offices by choosing from among the other 6 club members:
   $7 \cdot 6 \cdot 5 = 210$.
   Since these are two separate cases involving different initial assumptions, we find the total number of ways that the offices could be filled by adding the totals from these two cases:
   There are $30 + 210 = 240$ ways of filling the three offices under these conditions.

7. Suppose that 6 cards are drawn (at the same time) from a standard deck of 52 cards:

   (a) (5 points) How many different 6 card hands are possible?

   Notice that we do not care which order you receive the cards in, only which cards end up in your hand, so this is combination counting.
   Then there are $C(52, 6) = \frac{52!}{46!6!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47}{24} = 20,358,520$ different 6 card hands.

   (b) (5 points) How many 6 card hands have a four of a kind?

   Notice that this is similar to the example we did in class of having 4 of a kind (4 cards of the same "number"), except that there are now 6 cards in the hand. We will again start by choosing the type of card that occurs as our 4 of a kind. There are 13 options available. Notice that there is only one way to pick all four of these cards ($C(4, 4) = 1$).
   It remains to choose two additional cards to fill out the hand. I had it in mind that these cards should not match, but since this was not specifically stated in the problem, I gave full credit to either of these solutions: one in which any two of the remaining 48 cards are chosen to complete the hand, and then the slightly preferred solution which involves choosing two cards that do not form a pair to complete the hand.
   (a) [Pair allowed] there are $13 \cdot C(48, 2) = 13 \cdot \frac{48 \cdot 47}{2} = 13 \cdot 1128 = 14,664$ different 6 card hands with a 4 of a kind.

   (b) [Pair not allowed] there are $13 \cdot \frac{=}{1}3 \cdot \frac{48 \cdot 47}{2} - 12 \cdot C(4, 2) = 13 \cdot 1128 - 12 \cdot \frac{4 \cdot 3}{2} = 14,664 - 72 = 14,592$ different 6 card hands with a 4 of a kind (and without an additional pair).

   (c) (5 points) How many 6 card hands contain exactly 3 heart cards and exactly 3 club cards?

   This is a classic "double combination problem". There are 13 cards per suit, and we wish to choose three heart cards and three club cards.
   Then there are:
   $C(13, 3) \cdot C(13, 3) = \frac{13!}{10!3!} \cdot \frac{13!}{10!3!} = \frac{13 \cdot 12 \cdot 11}{6} \cdot \frac{13 \cdot 12 \cdot 11}{6} = 286 \cdot 286 = 81,796$ 6 card hands consisting of 3 heart cards and 3 club cards.

8. (3 points) Use roster notation to express the set:
   $A = \{ \ x \mid x$ is a letter that occurs both in the word *bread* **and** in the word *tree* $\}$

   Notice that we only want the letters that occur in both words, so $A = \{r, e\}$.

9. Determine whether or not the following sets are well defined:

   (a) (2 points) $J = \{ \ j \mid j$ is a joke that was told by Jimmy Fallon on October 1st. $\}$

   This set is well defined (a particular joke was either told by Jimmy Fallon that day or it wasn't).

   (b) (2 points) $W = \{ \ d \mid x$ is a day where the weather is cold. $\}$

   This set is not well defined (how do we decide whether or not a particular day is cold or not? This is subjective.)

10. (3 points each) Let $A = \{ \ x \mid x$ is an odd whole number less than 10 $\}$; $B = \{1, 2, 6, 7, 8\}$; $C = \{1, 3, 5, 7, 9\}$

    (a) Which (if any) of the three sets given above are equivalent? Justify your answer.

    Notice that as defined above, $A = \{1, 3, 5, 7, 9\}$. Then $n(A) = n(B) = n(C) = 5$, so all three of these sets are equivalent to each other.

    (b) Which (if any) of these sets above are equal? Justify your answer.

    Since, $A = \{1, 3, 5, 7, 9\}$. Then $A = C$. Notice that $6 \in B$ but $6 \notin A$ and $6 \notin C$, so $A \neq B$ and $B \neq C$.

11. (2 points each)Given that $A = \{ x \mid x \text{ is a letter in the word } \textit{blend} \}$, $B = \{ x \mid x \text{ is a letter in the word } \textit{blender} \}$, $C = \{ x \mid x \text{ is a letter in the word } \textit{blended} \}$, $D = \emptyset$, and $E = \{0, 2, \{0, 2\}\}$ indicate whether the following are **True** or **False** [you do NOT need to justify your answers]

(a) $n(C) = 7$

False.
Notice $n(C) = 5$.

(b) $A \subset B$

True
Compare the elements of these sets.

(c) $D \subset B$

True.
The empty set is a subset of *any* set.

(d) $A = C$

True.
Try writing both sets in roster notation.

(e) $0 \in E$

True.

(f) $\emptyset \in E$

False

(g) $\{0, 2\} \in E$

True.

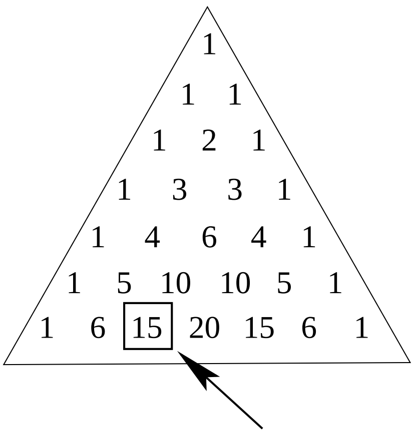
(h) $\{0, 2\} \subset E$

True.

12. (a) (3 points) How many subsets does the set $\{a, b, c, d, e, f, g, h\}$ have

Since there are 8 distinct elements in this set, this set has $2^8 = 256$ different subsets.

(b) (5 points) Use Pascal's Triangle to find the number of 2 element subsets of $\{a, b, c, d, e, f\}$



Since this set has 6 elements, we build down to the row in Pascal's Triangle corresponding to a set with six elements, and then look at the entry that corresponds with subsets of size 2. From this, we see that there are 15 subsets that contain exactly 2 elements.