Math 323
Exam 1 **Name:**__________________________________

Instructions: You will have 55 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.

1. Given the curve: $C = \begin{cases} x = t^2 - 1 & \text{for } t \in \mathbb{R} \\ y = t^4 \end{cases}$

   (a) (8 points) Find an *explicit equation* for an equation containing the graph of C in terms of x and y.

   Notice that $x + 1 = t^2$, so $(x+1)^2 = t^4$.

   Then $y = (x+1)^2$ or $y = x^2 + 2x + 1$ is an explicit equation relating x and y.

   (b) (8 points) Find the *equation of the tangent line* to this curve when $t = 2$.

   Recall that $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$. Also, $\frac{dy}{dt} = 4t^3$ and $\frac{dx}{dt} = 2t$.

   Then $\frac{dy}{dx} = \frac{4t^3}{2t} = 2t^2$. When $t = 2$, $m = 2(2)^2 = 8$.

   Also, when $t = 2$, $x = 3$ and $y = 16$.

   Therefore, the tangent line to the curve when $t = 2$ is given by: $y - 16 = 8(x - 3)$ or $y = 8x - 8$.

   (c) (8 points) Determine the *concavity* of the graph of C when $t = 2$.

   Recall that $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$. Also, $\frac{dy'}{dt} = 4t$ and $\frac{dx}{dt} = 2t$.

   Then $\frac{d^2y}{dx^2} = \frac{4t}{2t} = 2$, so when $t = 2$, $\frac{d^2y}{dx^2} = 2$. Therefore, $y = f(x)$ is concave up when $= 2$.

   (d) (8 points) Set up (But DO NOT evaluate) an integral with respect to t representing the arc length of C for $0 \le t \le 2$.

   Recall that $L = \displaystyle\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dy}\right)^2}\, dt$.

   Then $L = \displaystyle\int_0^2 \sqrt{(2t)^2 + (4t^3)^2}\, dt = \int_0^2 \sqrt{4t^2 + 16t^6}\, dt$.

2. (6 points) Find an equation *in rectangular coordinates* for the polar equation: $r^2 = 4r\cos\theta - 2r\sin\theta + 4$

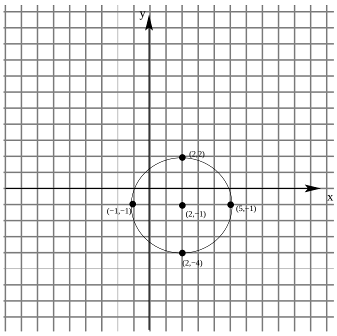
Recall that $x^2 + y^2 = r^2$, $x = r\cos\theta$, and $y = r\sin\theta$.

Then sustituting gives: $x^2 + y^2 = 4x - 2y + 4$.

Simplifying, we get $x^2 - 4x + y^2 + 2y = 4$, or $x^2 - 4x + 4 + y^2 + 2y + 1 = 4 + 1 + 4$.

Then $(x-2)^2 + (y+1)^2 = 9$.

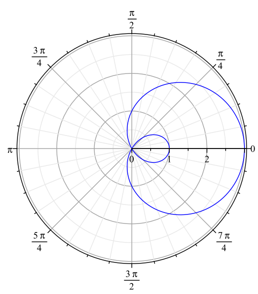
3. (6 points) **Graph** the resulting equation in rectangular coordinates.



4. (12 points) **Draw the graph** of the polar equation $r = 2\cos\theta - 1$. Be sure to indicate the orientation and *at least two points* on the curve.

We begin by costructing a table of values using multiples of $\frac{\pi}{2}$. Next, since we did not happen across the values where $r = 0$, we solve the equation: $0 = 2\cos\theta - 1$ or $\cos\theta = \frac{1}{2}$. Then $\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$.

θ	r
0	1
$\frac{\pi}{2}$	-1
π	$= -3$
$\frac{3\pi}{2}$	-1
2π	1
$\frac{\pi}{3}$	0
$\frac{5\pi}{3}$	0



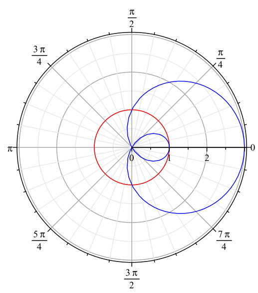
5. (12 points) Set up (But DO NOT evaluate) an integral representing the area inside $r = 2\cos\theta - 1$ and outside $r = 1$.

Notice that the points of intersection occur when $\theta = 0$, $\theta = \frac{\pi}{2}$, and when $\theta = \frac{3\pi}{2} = -\frac{\pi}{2}$.
Then, the area inside $r = 2\cos\theta - 1$ and outside $r = 1$ is given by:

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{2}\left[(2\cos\theta - 1)^2 - 1^2\right]\, d\theta$$

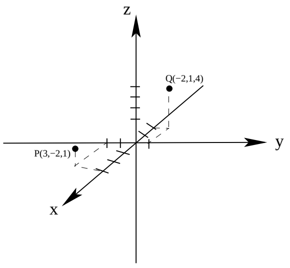
or

$$= \int_{\pi}^{\frac{3\pi}{2}} \left[(2\cos\theta - 1)^2 - 1\right]\, d\theta$$



6. Given the points: $P(3, -2, 1)$ and $Q(-2, 1, 4)$

   (a) (6 points) Plot P and Q in 3-space.



   (b) (6 points) Find $\overrightarrow{PQ}$ and $\|\overrightarrow{PQ}\|$

   $\overrightarrow{PQ} = \langle -2 - 3, 1 - (-2), 4 - 1\rangle = \langle -5, 3, 3\rangle$

   $\|\overrightarrow{PQ}\| = \sqrt{(-5)^2 + 3^2 + 3^2} = \sqrt{25 + 9 + 9} = \sqrt{43}$

   (c) (6 points) Find a unit vector in the **opposite** direction as $\overrightarrow{PQ}$.

   $$-\frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = -\frac{\sqrt{43}}{43} \cdot \langle -5, 3, 3\rangle = \langle \frac{5\sqrt{43}}{43}, -\frac{3\sqrt{43}}{43}, -\frac{3\sqrt{43}}{43}\rangle$$

   (d) (6 points) Find an equation for the sphere centered at P and containing Q.

   First notice that since $d(P, Q) = \sqrt{43}$, then $r = \sqrt{43}$. Also, $C = (3, -2, 1)$

   Then the equation for this sphere is given by $(x - 3)^2 + (y + 2)^2 + (z - 1)^2 = 43$

7. (10 points) Suppose the thrust of an airplane's engine produces a speed of 400mph and the velocity of the wind is given by $\langle 20, 60\rangle$. Find the direction the plane should head in order to fly due east (give your answer as a heading measured clockwise from due north). Also find the speed at which the plane travels its resulting course. [**Hint:** given the wind, should it be more than or less than 400mph?]

   Let $\vec{w} = \langle 20, 60\rangle$ be the wind vector, $\vec{p} = \langle x, y\rangle$ be the vector of the airplane in still air, and $\vec{p} + \vec{w} = \vec{r} = \langle s, 0\rangle$ be the resultant vector.

   Let θ be the angle between $\vec{p}$ and the positive x-axis.

   Since the speed of the airplane is 400 mph, then $x^2 + y^2 = 400^2$. Also, since the airplane ends up heading due east, then $y = -60$. Therefore, $x = \sqrt{400^2 - 60^2} = \sqrt{156,400} = 20\sqrt{391}$.

   Thus $\vec{p} = \langle 20\sqrt{391}, -60\rangle$, and so $\theta = \arctan \frac{-60}{20\sqrt{361}} \approx -8.63°$.

   Therefore, the airplane should travel at the heading 98.26° East of North.

   Finally, the speed of the plane $s = 20 + 20\sqrt{391} \approx 415.47$ mph.