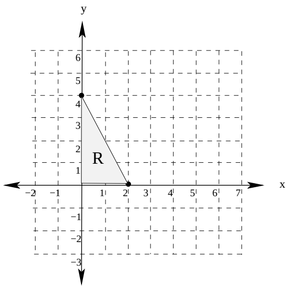
Math 323

Exam 4 - Practice Problems

1. Evaluate $\iint_R (y+x)\,dA$, where R is the region bounded by $x=0$, $y=0$, and $2x+y=4$.



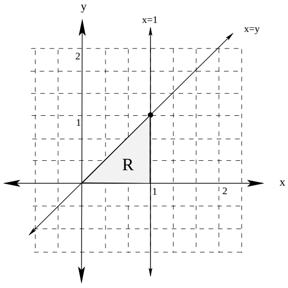
$$\int_0^2 \int_0^{4-2x} y+x\,dy\,dx = \int_0^2 \frac{1}{2}y^2 + xy\Big|_0^{4-2x} dx$$

$$= \int_0^2 \frac{1}{2}(4-2x)^2 + x(4-2x) - 0\,dx = \int_0^2 \frac{1}{2}(16-16x+4x^2) + 4x - 2x^2\,dx = \int_0^2 8-4x\,dx$$

$$= 8x - 2x^2\Big|_0^2 = 16-8 = 8$$

2. For each of the double integrals given below, first graph the region of integration. Then reverse the order of integration for the iterated integral and then evaluate the integral exactly.

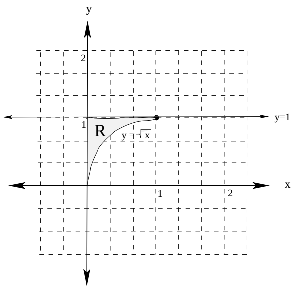
(a) $\displaystyle\int_0^1\int_y^1 3xe^{x^3}\,dx\,dy$



$$= \int_0^1\int_0^x 3xe^{x^3}\,dy\,dx = \int_0^1 3xye^{x^3}\bigg|_0^x\,dx = \int_0^1 3x^2e^{x^3}\,dx$$

$$= e^{x^3}\bigg|_0^1 = e^1 - e^0 = e - 1$$

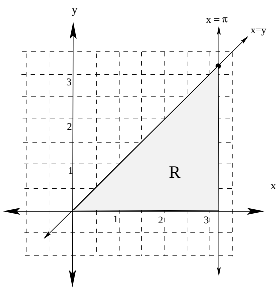
(b) $\displaystyle\int_0^1\int_{\sqrt{x}}^1 \frac{3}{4+y^3}\,dy\,dx$



$$= \int_0^1\int_0^{y^2} \frac{3}{4+y^3}\,dx\,dy = \int_0^1 \frac{3x}{4+y^3}\bigg|_0^{y^2}\,dy = \int_0^1 \frac{3y^2}{4+y^3}\,dy$$

$$= \ln|4+y^3|\bigg|_0^1 = \ln 5 - \ln 4 = \ln\frac{5}{4}$$

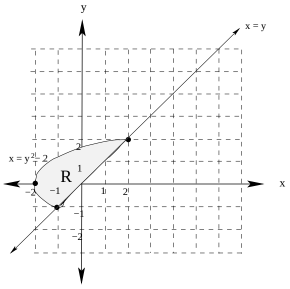
(c) $\displaystyle\int_0^\pi\int_y^\pi \left(1 - \frac{\cos x}{x}\right)\,dx\,dy$



$$= \int_0^\pi\int_0^x 1 - \frac{\cos x}{x}\,dy\,dx = \int_0^\pi y - y\frac{\cos x}{x}\bigg|_0^x\,dx = \int_0^\pi x - \cos x\,dx$$

$$= \frac{1}{2}x - \sin x\bigg|_0^\pi = \frac{\pi^2}{2} - 0 = \frac{\pi^2}{2}$$

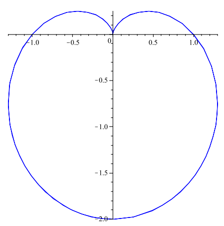
3. Express the volume of the solid bounded by the curves $z = x + 2, z = 0, x = y^2 - 2$ and $x = y$ as a double integral in rectangular coordinates. Also, sketch the region in the plane for the integration. DO NOT EVALUATE THE INTEGRAL.



Notice that the points of intersection for the region in the plane occur when $y = y^2 - 2$, or when $y^2 - y - 2 = 0$. That is, when $(y - 2)(y + 1) = 0$, so when $y = 2$ or $y = -1$. Therefore, the volume is given by:

$$\int_{-1}^{2}\int_{y^2-2}^{y} x + 2\,dx\,dy$$

4. Evaluate $\displaystyle\iint_R x\,dA$ where R is the region in the polar plane bounded by $r = 1 - \sin\theta$.



Converting everything to polar coodinates, $x = r\cos\theta$ and $r = 1 - \sin\theta$ is the cardioid graphed above.

Then we have $\displaystyle\int_0^{2\pi}\int_0^{1-\sin\theta} r\cos\theta r\,dr\,d\theta = \int_0^{2\pi}\int_0^{1-\sin\theta} r^2\cos\theta\,dr\,d\theta$

$$= \int_0^{2\pi} \frac{1}{3}r^2\cos\theta\Big|_0^{1-\sin\theta} d\theta = \frac{1}{3}\int_0^{2\pi}(1-\sin\theta)^3\cos\theta\,d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}(1 - 3\sin\theta + 3\sin^2\theta - \sin^3\theta)\cos\theta\,d\theta = \frac{1}{3}\left[\cos\theta - \frac{3}{2}\sin^2\theta + \sin^3\theta - \frac{1}{4}\sin^4\theta\Big|_0^{2\pi}\right] = 0$$

[Note that this is not a surprise as $f(x, y) = x$ is odd and the region R is symmetric with respect to the y-axis]

5. Calculate the mass of a lamina that occupies the plane region R bounded by the curve $(x - 1)^2 + y^2 = 1$ with density function $\rho(x, y) = \dfrac{1}{\sqrt{x^2 + y^2}}$.

Clearly, this problem is best represented in polar coordinates. First, $\rho(x, y) = \dfrac{1}{\sqrt{x^2 + y^2}}$ becomes $\rho(r, \theta) = \dfrac{1}{\sqrt{r^2}} = \dfrac{1}{r}$.

Next, the region in the plane is given by: $x^2 - 2x + 1 + y^2 = 1$, or $r^2 - 2r\cos\theta = 0$. Thus we have $r^2 = 2r\cos\theta$, or $r = 2\cos\theta$.

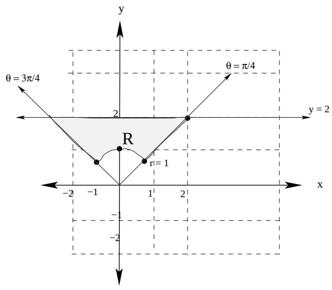
Therefore, using symmetry, we have mass$= 2\displaystyle\int_0^{\pi/2}\int_0^{\cos\theta} \frac{1}{r} r\,dr\,d\theta = 2\int_0^{\pi/2} 2\cos\theta d\theta$

$$= 4\sin\theta\Big|_0^{\frac{\pi}{2}} = 4(1) - 0 = 4$$

6. Sketch the region of integration for $\displaystyle\int_{\pi/4}^{3\pi/4}\int_{1}^{2/\sin\theta} r\,dr\,d\theta$.

Notice that we have $\dfrac{\pi}{4} \le \theta \le \dfrac{3\pi}{4}$. Also, $r = 1$ is the circle of radius 1 centered at the origin, and if $r = \dfrac{2}{\sin\theta}$, then $r\sin\theta = 2$ or $y = 2$.

Therefore the region of integration is as follows:



7. Find the surface area of the surface S where S is first octant portion of the hyperbolic paraboloid $z = x^2 - y^2$ that is inside the cylinder $x^2 + y^2 = 1$.

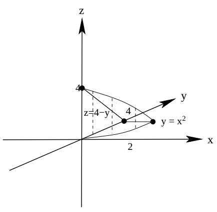
Recall that $S = \displaystyle\iint_R \sqrt{f_x^2 + f_y^2 + 1}\,dA$. Here, $z = f(x, y) = x^2 - y^2$, so $f_x = 2x$ and $f_y = -2x$

So we have $S = \displaystyle\iint_R \sqrt{(2x)^2 + (-2y)^2 + 1}\,dA = \iint_R \sqrt{4x^2 + 4y^2 + 1}\,dA$ where R is portion of the cylinder $x^2 + y^2 = 1$ in the first octant (so $x \ge 0$, $y \ge 0$, and $z \ge 0$).

Converting to polar coordinates gives: $\displaystyle\int_0^{\frac{\pi}{4}}\int_0^1 \sqrt{4r^2+1}r\,dr\,d\theta = \int_0^{\frac{\pi}{4}} \frac{2}{3}\cdot\frac{1}{8}(4r^2+1)^{\frac{3}{2}}\Big|_0^1\,d\theta$

$\displaystyle= \frac{1}{12}\int_0^{\frac{\pi}{4}} 5^{\frac{3}{2}} - 1\,d\theta = \frac{1}{12}\frac{\pi}{4}\left(5^{\frac{3}{2}} - 1\right) = \frac{\pi}{48}\left(5^{\frac{3}{2}} - 1\right).$

8. Let $I = \displaystyle\int_0^2\int_{x^2}^4\int_0^{4-y} x + yz\,dz\,dy\,dx$. Sketch the solid Q over which the iterated integral takes place, and rewrite the iterated integral in the order $dx\,dz\,dy$. DO NOT EVALUATE THE INTEGRAL.



Notice that we have $0 \le z \le 4 - y$, $x^2 \le y \le 4$, and $0 \le x \le 2$

Then we also have $0 \le x \le \sqrt{y}$, $0 \le z \le 4 - y$, and $0 \le y \le 4$

Thus the integral in the required order is: $\displaystyle\int_0^4\int_0^{4-y}\int_0^{\sqrt{y}} x + yz\,dx\,dz\,dy$

9. Find the mass of the cylinder of radius 3 between $z = 0$ and $z = 4$ if the density at the point (x, y, z) is given by $\delta(x, y, z) = z + \sqrt{x^2 + y^2}$.

This mass integral is best represented in cylindrical coordinates:

$$m = \int_0^{2\pi}\int_0^3\int_0^4 (z + r)r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^3\int_0^4 zr + r^2\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 \frac{1}{2}z^2r + r^2z\Big|_0^4\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^3 8r + 4r^2dr\,d\theta$$

$$= \int_0^{2\pi} 4r^2 + \frac{4}{3}r^3\Big|_0^3\,d\theta = \int_0^{2\pi} 36 + 36\,d\theta = 72\theta\Big|_0^{2\pi} = 144\pi$$

10. Let Q be the tetrahedron bounded by the coordinate planes and the plane $2x+5y+z=10$. Find the mass and center of mass of Q is the density at a point $P(x,y,z)$ is directly proportional to the distance from P to the xz-plane.

First, notice that the distance of a point from the xz plane is given by $|y|$. Since the volume Q is a subset of the first octant y is always positive, thus $\delta(x,y,z)=ky$ for some constant of proportionality k.

Next, notice that if we decompose Q with respect to z, then the top surface is given by $z=10-2x-5y$ and the bottom surface is given by $z=0$.

The region R in the plane that deterines the outer pair of limits of integration in our triple integrals is a triangle in the first quadrant of the xy-plane. This triangle is bounded above by $y=\frac{1}{5}(10-2x)$ [set $z=0$ in the equation for the plane and then solve for y] and below by $y=0$.

Finally, we see that $0\le x\le 5$ [notice tha x-intercept of the plane is at $x=5$.]

From this, we can set up triple integrals for the mass and for each of the three moments:

$$m=\int_0^5\int_0^{\frac{1}{5}(10-2x)}\int_0^{10-2x-5y} ky\,dz\,dy\,dx=\int_0^5\int_0^{\frac{1}{5}(10-2x)} kyz\Big|_0^{10-2x-5y}dy\,dx=\int_0^5\int_0^{\frac{1}{5}(10-2x)} ky(10-2x-5y)\Big|_0^{10-2x-5y}dy\,dx$$

$$=k\int_0^5\int_0^{\frac{1}{5}(10-2x)} 10y-2xy-5y^2\,dy\,dx=k\int_0^5 5y^2-xy^2-\frac{5}{3}y^3\Big|_0^{\frac{1}{5}(10-2x)}dx$$

$$=k\int_0^5 5\left(\frac{10-2x}{5}\right)^2-x\left(\frac{10-2x}{5}\right)^2-\frac{5}{3}\left(\frac{10-2x}{5}\right)^3dx=k\int_0^5\left(\frac{10-2x}{5}\right)^2\left[5-x-\frac{5}{3}\left(\frac{10-2x}{5}\right)\right]dx$$

$$=k\int_0^5\left(\frac{10-2x}{5}\right)^2\left[\frac{5-x}{3}\right]dx=k\int_0^5\frac{4}{75}(5-x)^3\,dx=k\frac{1}{75}(5-x)^4\Big|_0^5=k\frac{625}{75}=\frac{25}{3}k$$

Similarly, (omitting the details of the evaluations, which are very much like the one above):

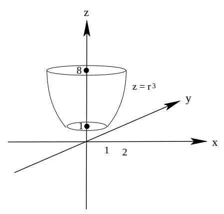
$$M_{yz}=\int_0^5\int_0^{\frac{1}{5}(10-2x)}\int_0^{10-2x-5y} xky\,dz\,dy\,dx=\frac{25}{3}k$$

$$M_{xz}=\int_0^5\int_0^{\frac{1}{5}(10-2x)}\int_0^{10-2x-5y} yky\,dz\,dy\,dx=\frac{20}{3}k$$

$$M_{xy}=\int_0^5\int_0^{\frac{1}{5}(10-2x)}\int_0^{10-2x-5y} zky\,dz\,dy\,dx=\frac{50}{3}k$$

Thus $\overline{x}=\frac{M_{yz}}{m}=1$, $\overline{y}=\frac{M_{xz}}{m}=\frac{4}{5}$, and $\overline{z}=\frac{M_{xy}}{m}=2$.

11. Let Q be the region between $z=(x^2+y^2)^{3/2}$ and $z=1$, and inside $x^2+y^2=4$. Sketch the region Q, and then write $\iiint_Q\sqrt{x^2+y^2}e^z\,dV$ as an integral in the best(for this example) 3-dimensional coordinate system. DO NOT EVALUATE THE INTEGRAL.



Given the form of the solid region and the function, cylindrical coordiates is the best system to use to express this integral.

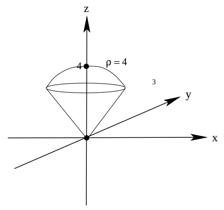
Converting the top and bottom surfaces, we have $z=(r^2)^{\frac{3}{2}}=r^3$ and $z=1$ [this was a typo on the original handout-sorry!]

Notice that the intersection between these two surfaces is: $(x^2+y^2)^{3/2}=1$, or $x^2+y^2=1$, the circle of radius 1 about the origin.

The region in the plane is all points between the circle of radius one and the circle of radius 2 both centered at the origin, and the integrand becomes re^z.

Thus the integral is: $\displaystyle\int_0^{2\pi}\int_1^2\int_1^{r^3}(re^z)\,r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_1^2\int_1^{r^3}r^2e^z\,dz\,dr\,d\theta$

12. Set up a triple integral in spherical coordinates for the volume V of the region between $z = \sqrt{3x^2 + 3y^2}$ and the sphere $x^2 + y^2 + z^2 = 16$. Be sure to include a sketch of the region, and DO NOT EVALUATE THE INTEGRAL.



Translating $z = \sqrt{3x^2 + 3y^2}$, we have $z^2 = 3x^2 + 3y^2$, or $4z^2 = 3x^2 + 3y^2 + 3z^2 = 3\rho^2$

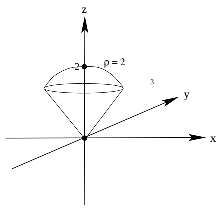
Then $4(\rho\cos\phi)^2 = 3\rho^2$, or $4\rho^2\cos^2\phi = 3\rho^2$. Thus we have $4\cos^2\phi = 3$, or $\cos^2\phi = \frac{3}{4}$.

Hence $\cos\phi = \pm\frac{\sqrt{3}}{2}$. Since we only want values between 0 and π, we take $\phi = \frac{\pi}{6}$, [a 30° cone].

Notice that $x^2 + y^2 + z^2 = 16$ is the sphere of readius 4 about the origin, and has the form $\rho = 4$.

Therefore, the integral has the form $\int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^4 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

13. (a) Let Q be the region between $z = \sqrt{3x^2 + 3y^2}$ and $z = \sqrt{4 - x^2 - y^2}$. Sketch the region Q



(b) Set up, but do not evaluate a triple integral in rectangular coordinates that gives the volume of Q.

Using symmetry, $\iiint_Q 1\, dV = 4\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{3x^2+3y^2}}^{\sqrt{4-x^2-y^2}} 1\, dz\, dy\, dx$

(c) Set up, but do not evaluate a triple integral in cylindrical coordinates that gives the volume of Q.

Again using symmetry, $\iiint_Q 1\, dV = 4\int_0^{\frac{\pi}{2}} \int_0^1 \int_{\sqrt{3}r}^{\sqrt{4-r^2}} 1\, r\, dz\, dr\, d\theta$

(d) Set up, but do not evaluate a triple integral in spherical coordinates that gives the volume of Q.

We need to figure out the angle of declination for the cone. Starting with $z^2 = 3x^2 + 3y^2$, we see that $4z^2 = 3x^2 + 3y^2 + 3z^2$, or $4z^2 = 3\rho^2$.

Therefore, since $z = \rho\cos\phi$, $4\rho^2\cos^2\phi = 3\rho^2$. Hence $4\cos^2\phi = 3$, or $\cos^2\phi = \dfrac{3}{4}$.

Thus $\cos\phi = \pm\dfrac{\sqrt{3}}{2}$. The solution to this in the first quadrant is $\phi = \dfrac{\pi}{6}$.

So we have $\iiint_Q 1\, dV = \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

(e) Pick one of the triple integrals you found above and evaluate it in order to find the volume of Q exactly.

We will use the spherical integral: $\int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{6}} \frac{1}{3}\rho^3 \sin\phi \Big|_0^2 \, d\phi \, d\theta$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{6}} \frac{8}{3}\sin\phi \, d\phi \, d\theta = \frac{8}{3}\int_0^{2\pi} -\cos\phi \Big|_0^{\frac{\pi}{6}} d\theta$$

$$\frac{8}{3}\int_0^{2\pi} -\cos\phi \Big|_0^{\frac{\pi}{6}} d\theta = \frac{8}{3}\int_0^{2\pi} -\left(\frac{\sqrt{3}}{2} - 1\right) d\theta = \frac{8}{3}\left(\frac{1-\sqrt{3}}{2}\right)\theta\Big|_0^{2\pi} = \frac{8}{3}(2\pi)\left(1 - \frac{\sqrt{3}}{2}\right) = \frac{16\pi}{3}\left(1 - \frac{\sqrt{3}}{2}\right)$$