Math 323
LaGrange Multipliers

Example: Consider the line $y = 5 - 3x$. What is the point on this line closest to the origin? There are several methods that can be used to solve this problem.

The calculus I method would be to derive a function that gives the distance of a point on the line $y = 5 - 3x$ from the origin as a function of x and then optimize this function.

A more elegant way to solve this is to notice that the circle of radius 1 centered at the origin does not intersect the line $y = 5 - 3x$ while the circle of radius 3 does.

In fact, there is some perfect radius $0 < r < 3$ for which the circle of radius r centered at the origin is tangent to the line $y = 5 - 3x$, and the closest point on the line to the origin is the point of tangency.

With this in mind, we consider $f(x, y) = x^2 + y^2$, and let $g(x, y) = 3x + y - 5 = 0$ (we just rearranged $y = 5 - 3x$). We are looking for the level curve of f that is tangent to $g(x, y) = 0$.

That is, a level curve for which the tangent line to f is parallel to the line $y = 5 - 3x$, or a point where ∇f and ∇g are parallel to one another, or a point where $\nabla f = \lambda \nabla g$ for some constant λ.

Now, $\nabla f = \langle 2x, 2y\rangle$ and $\nabla g = \langle 3, 1\rangle$, so $\langle 2x, 2y\rangle = \lambda\langle 3, 1\rangle$

Thus $2x = \lambda \cdot 3$ and $2y = \lambda \cdot 1$, or $\frac{2}{3}x = \lambda = 2y$, so $2x = 6y$, or $x = 3y$.

Substituting this into $g(x, y) = 3x + y - 5 = 0$, we have $9y + y - 5 = 0$, or $10y = 5$, so $y = \frac{1}{2}$, and $x = \frac{3}{2}$.

Hence the point on the line $y = 5 - 3x$ that is closest to the origin is $(\frac{3}{2}, \frac{1}{2})$.

This example is an illustration of a much more general principle.

La Grange's Theorem: Suppose f and g are functions of two variables with continuous first partial derivatives and suppose that $\nabla g \neq \vec{0}$ throughout a region of the xy-plane. If f has an extremum $f(x_0, y_0)$ subject to the constraint $g(x, y) = 0$, then there is a real number λ such that $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$

Proof Sketch: Since the graph of $g(x, y) = 0$ is a curve $\mathcal{C}$ in the plane and g has continuous first partials, then $\mathcal{C}$ has parameterization

$$\mathcal{C}: \begin{cases} x = h(t) \\ y = k(t) & t \in I \end{cases}$$ where $h(t)$ and $k(t)$ are continuous functions on some interval I.

Let $\vec{r}(t) = \langle h(t), k(t)\rangle$ be the associated vector-valued function. Let t_0 be the value in I such that $h(t_0) = x_0$ and $k(t_0) = y_0$ and let $F(h(t), k(t))$ be the composite function. Since $F(t_0)$ in an extremum on a region with continuous partials, $F'(t_0) = 0$.

By the Chain Rule, $F'(t) = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = f_x h'(t) + f_y k'(t)$.

Therefore, when $t = t_0$, we have: $0 = F'(t_0) = f_x(x_0, y_0)h'(t_0) + f_y(x_0, y_0)k'(t_0) = \nabla f(x_0, y_0) \cdot \vec{r}'(t_0)$

Hence $\nabla f(x_0, y_0)$ is orthogonal to the tangent vector $\vec{r}'(t_0)$. Moreover, $\nabla g(x_0, y_0)$ is also orthogonal to $\vec{r}'(t_0)$ since $\mathcal{C}$ is a level curve of g.

Therefore $\nabla f(x_0, y_0)$ is parallel to $\nabla g(x_0, y_0)$. That is, $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some real number λ.

We call λ a **La Grange Multiplier**.

Corollary: The points at which a function f of two variables has relative extrema subject to the constraint $g(x, y) = 0$ are included among the points determined by the first two coordinates of the solutions (x, y, λ) to the system of equations:

$$\mathcal{C}: \begin{cases} f_x(x, y) = \lambda g_x(x, y) \\ f_y(x, y) = \lambda g_y(x, y) \\ g(x, y) = 0 \end{cases}$$

Corollary: [3 variable version] The points at which a function f of three variables has relative extrema subject to the constraint $g(x, y, z) = 0$ are included among the points determined by the first three coordinates of the solutions (x, y, z, λ) to the system of equations:

$$\mathcal{C}: \begin{cases} f_x(x, y, z) = \lambda g_x(x, y, z) \\ f_y(x, y, z) = \lambda g_y(x, y, z) \\ f_z(x, y, z) = \lambda g_z(x, y, z) \\ g(x, y, z) = 0 \end{cases}$$

Examples:

1. Suppose that we want to cut a rectangular bean from a circular log of radius 1 foot. What dimensions will maximize the cross-sectional area of the beam?

   We set up a coordinate system for a cross section of the log by looking at the unit circle centered at the origin. We can specify the dimensions of a beam cut from this log by selecting a point in the first quadrant. The x-coordinate gives half the width of the beam, and the y-coordinate gives half the height of the beam. Then $A = f(x, y) = (2x)(2y) = 4xy$, subject to the constraint $g(x, y) = x^2 - y^2 - 1 = 0$ (we assume the the maximum cross-sectional area occurs when we cut the log so that the "corners" of the beam lie along the circumference of the log's cross-section).

   First notice that the partial derivatives of f and g are: $f_x = 4y$, $f_y = 4x$, $g_x = 2x$, and $g_y = 2y$.

   Therefore, by La Grange's Theorem, we are looking for points on the circle satisfying $4y = \lambda(2x)$ and $4x = \lambda(2y)$

   That is, $\lambda = \dfrac{4y}{2x} = \dfrac{4x}{2y}$, so $8y^2 = 8x^2$ or $x^2 = y^2$.

   Substituting this into the constraint equation gives $x^2 + x^2 = 2x^2 = 1$, so $x^2 = \frac{1}{2}$ and $x = \pm\frac{\sqrt{2}}{2}$ and $y = \pm\frac{\sqrt{2}}{2}$

   Recall that we can assume the the point determining the dimensions of the beam is in the first quadrant, so the point is $P(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and hence the dimensions of the beam are $(\sqrt{2}, \sqrt{2})$.

2. Suppose that a rectangular box with no lid is to be constructed from $12m^2$ of cardboard. Find the maximum volume of such a box.

   We set up a coordinate system for the box by letting at the x-coordinate give the width of the box, the y-coordinate give the length of the box, and the z coordinate give the height of the box. Then $V = f(x, y, z) = xyz$, subject to the constraint that the total surface area of the box satisfies: $g(x, y, z) = 2xz + 2yz + xy - 12 = 0$.

   First notice that the partial derivatives of f and g are: $f_x = yz$, $f_y = xz$, $f_z = xy$, $g_x = 2z + y$, $g_y = 2z + x$, and $g_z = 2x + 2y$.

   Therefore, by La Grange's Theorem, we are looking for points satisfying $yz = \lambda(2z + y)$, $xz = \lambda(2z + x)$, and $xy = \lambda(2x + 2y)$

   That is, (multiplying by each "missing" variable: $xyz = \lambda(2xz + xy)$, $xyz = \lambda(2yz + xy)$, and $xyz = \lambda(2xz + 2yz)$

   Equating the first two gives: $\lambda(2xz + xy) = \lambda(2yz + xy)$, or $2xz + xy = 2yz + xy$.

   Thus $2xz = 2yz$, so either $x = y$ or $z = 0$.

   Equating the last two gives: $\lambda(2yz + xy) = \lambda(2xz + 2yz)$, or $2yz + xy = 2xz + 2yz$.

   Thus $xy = 2xz$, so either $x = 0$ or $y = 2z$.

   Since we clearly do not want a box with no width or no height, the box of maximal volume must satisfy $x = y = 2z$.

   Substituting this into the constraint equation gives $2(2z)z + 2(2z)z + (2z)(2z) - 12 = 0$ or $4z^2 + 4z^2 + 4z^2 = 12$, so $12z^2 = 12$ and hence $z = \pm 1$.

   We reject the negative solution and conclude that $x = y = 2$ and $z = 1$ gives the width, length, and height of the box with maximal volume.

Two Constraint Optimization: Let $f(x, y, z)$ be a function subject to *two* constraints $g(x, y, z) = 0$ and $h(x, y, z) = 0$. If an extremum of f subject to these constraints occurs at a point $P(x_0, y_0, z_0)$ where $\nabla g(x_0, y_0, z_0)$ and $\nabla h(x_0, y_0, z_0)$ are non-zero and non-parallel, then $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$

Example: The plane $x + y + z = 12$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the lowest and highest points on this ellipse.

Notice that the two constraint system that can be used to find the highest and lowest points in this intersection is:
$f(x, y, z) = z$, $g(x, y, z) = x + y + z - 12 = 0$, and $h(x, y, z) = x^2 - y^2 - z = 0$.