Math 323
Exam 2 - Projectile Motion

Basic Laws of Motion:

Let $\vec{r}(t)$ be a vector valued function that gives the position vector of an object as a function of time t.

Then $\vec{v}(t) = \vec{r}\,'(t)$ gives the velocity vector of the object as a function of time t, and $\|\vec{v}(t)\|$ gives the speed of the object at time t.

Similarly, $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$ gives the acceleration vector of the object as a function of time t.

Note: Given any one of $\vec{r}(t)$, $\vec{v}(t)$, and $\vec{a}(t)$ (along with enough initial conditions) we can use differentiation and/or integration to solve for the other two related functions.

Example 1: A 2D Projectile Motion Problem.
Suppose that a projectile is launched with initial velocity $v_0 = 150$ ft/s from a height of 0 feet and at an angle of $\theta = \frac{\pi}{3}$. Assuming that the only force acting on the object is gravity, find the maximum altitude, horizontal range, and speed at impact of this projectile.

Solution: Recall that the downward force of gravity is $32ft/s^2$. Since the only forces acting on this projectile are gravity and its initial velocity, we can model this situation in 2 dimensions. We take $\vec{j}$ to be the unit vector in the upward direction.

Then $\vec{a}(t) = -32\vec{j}$, so, integrating: $\vec{v}(t) = 0\vec{i} - -32\vec{j} + \vec{C}$.

Now, using trigonometry, $\vec{v}(0) = 150\cos\frac{\pi}{3}\vec{i} + 150\sin\frac{\pi}{3}\vec{j} = 75\vec{i} + 75\sqrt{3}\vec{j}$.
Thus $\vec{v}(t) = 75\vec{i} + (75\sqrt{3} - 32t)\vec{j}$.

Also notice that $\vec{r}(0) = \vec{0}$, so, integrating $\vec{v}(t)$, we see that $\vec{r}(t) = 75t\vec{i} + (75\sqrt{3}t - 16t^2)\vec{j}$.

The maximum altitude occurs when the vertical component of $\vec{v}(t)$ is zero, that is, when $75\sqrt{3} - 32t = 0$, or when $t = \frac{75\sqrt{3}}{32}$.
Therefore, the maximum altitude is found by looking at the vertical position at that time:
$75\sqrt{3}\left(\frac{75\sqrt{3}}{32}\right) - 16\left(\frac{75\sqrt{3}}{32}\right)^2 \approx 263.67$ feet.

The horizontal range is found by first finding the time when the projectile hits the ground, and then finding the horizontal component of position at that time:

The projectile hits the ground when the vertical component of the position function is zero. That is, when $75\sqrt{3}t - 16t^2 = 0$.
Algebraically, this has solutions when $t = 0$ or $75\sqrt{3} - 16t = 0$. Since $t = 0$ is the time before the projectile leaves the ground, the projectile returns to the ground when $t = \frac{75\sqrt{3}}{16}$.
Then the range is: $75\left(\frac{75\sqrt{3}}{16}\right) \approx 608.92$ feet.

The speed at impact is found by looking at the magnitude of the velocity vector at the time of impact:
$\|\vec{v}\left(\frac{75\sqrt{3}}{16}\right)\| = \sqrt{(75)^2 + \left(75\sqrt{3} - 150\right)^2} = \sqrt{75^2 + \left(-75\sqrt{3}\right)^2} = 150feet/sec.$

Notice that since this projectile was launched from the ground, we should not be surprised that the time it takes to reach the ground it twice the amount of time that it took for it to reach its maximum height. We also should not be surprised that the velocity when the projectile hits the ground is the same as its launch velocity but in the opposite direction.

Note: This sort of symmetry will **not** occur when a projectile of launched from a height above ground level.

Example 2: A 3D Projectile Motion Problem.

Suppose that a projectile of mass 1kg is launched from the ground due north with an initial velocity of 100 m/s and at an angle of $\frac{\pi}{4}$. Suppose that the only forces acting on the projectile are the downward force of gravity and a wind blowing West with a constant force of 5 Newtons.

We must use a 3D coordinate system (let the positive x-axis be East, the positive y-axis North, and let positive z be Up). Then we have:
$\vec{w} = -5\vec{\imath}$ and $\vec{g} = -9.8\vec{k}$, so $\vec{a}(t) = -5\vec{\imath} + 0\vec{\jmath} - 9.8\vec{k}$.

As in the previous example, using basic trigonometry, $\vec{v}(0) = 0\vec{\imath} + 100 \cdot \frac{\sqrt{2}}{2}\vec{\jmath} + 100 \cdot \frac{\sqrt{2}}{2}\vec{k} = 0\vec{\imath} + 50\sqrt{2}\vec{\jmath} + 50\sqrt{2}\vec{k}$.
Then $\vec{v}(t) = (-5t\vec{\imath} + 50\sqrt{2}\vec{\jmath} - 9.8t + 50\sqrt{2}\vec{k})$

Notice that we set up our coordinate system so that $\vec{r}(0) = \vec{0}$. Therefore, $\vec{r}(t) = -\frac{5}{2}t\vec{\imath} + 50\sqrt{2}t\vec{\jmath} + \left(-4.9t^2 + 50\sqrt{2}t\right)\vec{k}$.

The projectile lands when the $\vec{k}$ component of the position function is 0. That is, when $-4.9t^2 + 50\sqrt{2}t = 0$. This has solutions when $t = 0$, which corresponds to the time that the projectile is launched, and when $-4.9t + 50\sqrt{2} = 0$, or when $t = \frac{50\sqrt{2}}{4.9}$.

Thus the final position of the projectile is: $\vec{r}(\frac{50\sqrt{2}}{4.9}) = \langle -\frac{5}{2}\left(\frac{50\sqrt{2}}{4.9}\right)^2, 50\sqrt{2}\left(\frac{50\sqrt{2}}{4.9}\right), 0\rangle \approx \langle -520.62, 1020.4, 0\rangle$ where the coordinates are in meters.