Math 450
Exam 1 Solutions

**Instructions:** You will have 75 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.
Work your exam on separate sheets of paper. Be sure to put your name on at least the front page.

1. Given that the Maclaurin Series for arctan is given by:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$$

   (a) (8 points) Find the number of terms necessary to approximate arctan(1) to within $10^{-5}$.

   Using the Alternating Series Test, we need to find $n$ such that $Error \leq \left|(-1)^n \frac{x^{2n+1}}{2n+1}\right| \leq \frac{1}{2n+1} < 10^{-5}$.
   That is, $\frac{1}{10^{-5}} < 2n+1$, so we need $n > \frac{100{,}000-1}{2} = 49{,}999.5$, or $n = 50{,}000$.

   (b) (6 points) Use the first 6 non-zero terms of this series to approximate arctan(1). From this, find an approximation of $\pi$.

   $\arctan(1) \approx 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{11} - \frac{1}{13} \approx .744011544$.
   From this, since $\arctan(1) = \frac{\pi}{4}$, $\pi \approx 4(.744011544) = 2.976046176$.

   (c) (6 points) Find the absolute error and the relative error for the estimate of $\pi$ you found in part (b).

   The absolute error is given by: $|\pi - 2.976046176| \approx 0.165546478$.
   The relative error is given by: $\frac{|\pi - 2.976046176|}{\pi} \approx 0.05269508$.

2. Let $f(x) = x^2 + x - 5$.

   (a) (5 points) Prove that $f(x)$ has a root in $[1, 2]$

   Notice that $f(x)$ is continuous since it is a polynomial. Also, $f(1) = 1 + 1 - 5 = -3$ and $f(2) = 4 + 2 - 5 = 1$. Therefore, by the intermediate value theorem, for some $c$ in $(1, 2)$, $f(c) = 0$.

   (b) (7 points) Let $p_0 = 1.5$. Determine $p_1$ and $p_2$ using the Bisection Method.

   $f(1.5) = -1.25$, so $p_1 = \frac{1.5+2}{2} = 1.75$.
   $f(1.75) = -0.1875$, so $p_2 = \frac{1.75+2}{2} = 1.875$.

   (c) (12 points) Find a function $g(x)$ that could be used as a fixed point iteration formula to find a root of $f(x)$ on $[1, 2]$. Make sure to verify that the hypotheses of Theorem 2.3 hold for your $g(x)$.

   Consider the equation $x^2 + x - 5 = 0$. Then $x^2 = 5 - x$, or $x = \sqrt{5-x}$ (we take the positive square root since we are looking at the interval $[1, 2]$). We will verify that $g(x) = \sqrt{5-x}$ satisfies the hypotheses of Theorem 2.3.
   First notice that $g'(x) = \frac{1}{2}\frac{-1}{\sqrt{5-x}} = \frac{-1}{2\sqrt{5-x}}$. Since $g'(x) < 0$ on $[1, 2]$, $g(x)$ is decreasing on this interval.
   Also, $g(1) = 2$ and $g(2) = \sqrt{3} \approx 1.732$, therefore, $g([1, 2]) \subseteq [1, 2]$.
   Furthermore, since $g''(x) = \frac{-1}{4(5-x)^{\frac{3}{2}}} < 0$ on $[1, 2]$, then $g'(x)$ is also decreasing on the interval $[1, 2]$. Moreover, $g'(1) = -\frac{1}{4} = -0.25$ and $g'(2) \approx -0.2886$. Therefore, if we take $k = -0.29$, the $|g'(x)| < k = 0.29 < 1$ for all $x \in [1, 2]$.
   Thus the hypotheses of Theorem 2.3 are satisfied, and $g(x)$ can be used as the basis for a fixed point iteration formula to find a root of $f(x)$ on $[1, 2]$.

   (d) (7 points) Find an upper bound on the number of iterations that would be needed to obtain an approximation of a root to within $10^{-4}$ when using Fixed Point Iteration with $p_0 = 2$ and the $g(x)$ you found above.

   Recall that the error when using fixed point iteration satisfies: $Error = |pn - p| \leq k^n(1)$. Notice that if $(.29)^n = .0001$, then $n\log(0.29) = -4$, so $n = \frac{-4}{\log(0.29)} \approx 7.44$.

Thus $n = 8$ is an upper bound in the number of iterations needed.
**Note:** We can also get an upper bound by using the fact that $|p_n - p| \le \frac{k^n}{1-k}|p_0 - p_1|$ (This method gives a slightly better bound of $n = 7$).

(e) (7 points) Let $p_0 = 2$ with $g(x)$ as above. Use fixed point iteration to compute $p_1$, $p_2$, and $p_3$.

Using $g(x) = \sqrt{5-x}$ as above, we see that:
$p_1 = g(2) \approx 1.732050808$, $p_2 = g(g(2)) = g(p_1) \approx 1.807746993$, and $p_3 = g(g(g(2))) = g(p_2) \approx 1.786687719$

(f) (7 points) Let $p_0 = 2$. Use Newton's Method on $f(x)$ to compute $p_1$ and $p_2$.

Recall that in Newton's Method, we have $p_{n+1} = p_n - \frac{f(p_n)}{f'(p_0)}$. Here, since $f(x) = x^2 + x - 5$, then $f'(x) = 2x + 1$, and our Newton's met hod expression is given by: $g(x) = x - \frac{x^2+x-5}{2x+1}$.

From this, given $p_0 = 2$, we see that $p_1 = g(2) \approx 1.800000000$, and $p_2 = g(g(2)) \approx 1.791304348$

(g) (7 points) Let $p_0 = 1$ and $p_1 = 2$. Use the Method of False Position to compute $p_2$ and $p_3$.

Recall that the method of False position is a modification of the Secant Method in which a sign testing check is added to ensure that the hypotheses of the Intermediate Value Theorem continue to be satisfied throughout the algorithm.
Notice that $f(p_0) = f(1) = -3 < 0$, and $f(p_1) = f(2) = 1 > 0$.
Using the secant method, $p_2 = p_1 - \frac{f(p_1)[p_1 - p_0]}{f(p_1) - f(p_0)} = 2 - \frac{1[2-1]}{1-(-3)} = 1.75$.
Notice that $f(1.75) = -0.1875 < 0$, so we proceed using $p_1 = 2$ and $p_2 = 1.75$.
Then $p_3 = p_2 - \frac{f(p_2)[p_2 - p_1]}{f(p_2) - f(p_1)} = 1.75 - \frac{(-0.1875)[1.75-2]}{-0.1875-1} approx 1.789473684$.

3. (8 points) Derive the iteration formula used in Newton's Method (you do not need to prove that it converges).

Notice that given a differentiable function $f(x)$, if we consider the tangent line to $f$ at the point $(p_0, f(p_0))$, the tangent line satisfies the following equation:
$y - f(p_0) = f'(p_0)(x - p_0)$.

In Newton's method, given an initial approximation $p_0$, our next approximation is given by the $x$-coordinate of the $x$-intercept of the tangent line to $f(x)$ at the point $(p_0, f(p_0))$. To find this $x$-coordinate, we set $y = 0$ and $x = p_1$ in the previous equation. This gives:
$0 - f(p_0) = f'(p_0)(p_1 - p_0)$, or $-f(p_0) = f'(p_0)(p_1 - p_0)$
Then $-f(p_0) = f'(p_0)(p_1) - f'(p_0)(p_0)$, or $f'(p_0)(p_0) - f(p_0) = f'(p_0)(p_1)$
Thus $p_1 = \frac{f'(p_0)(p_0) - f(p_0)}{f'(p_0)}$, or $p_1 = p_0 - \frac{f(p_0)}{f'(p_0)}$.

Applying this recursively given the formula: $p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$ for $n \ge 1$.

4. (12 points) Prove the following:

Assume $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$. Suppose, in addition, that $g'$ exists on $(a,b)$ with $|g'(x)| \le k < 1$ for all $x \in (a,b)$. Prove that if $p_0 \in [a,b]$, then the sequence defined by $p_n = g(p_n - 1)$, $n \ge 1$, converges to the unique fixed point in $[a,b]$. [**Note:** For this proof, you may assume that Theorem 2.2 holds – that is that we know that there is a unique fixed point in this interval.]

By Theorem 2.2, there is a unique fixed point $p \in [a,b]$. Since $g(x) \in [a,b]$ for all $x \in [a,b]$, Given that $p_0 \in [a,b]$, we see that $g(p_0) = p_1 \in [a,b]$, and if $p_n \in [a,b]$, then $g(p_n) = p_{n+1} \in [a,b]$. That is, $p_n \in [a,b]$ for all $n \ge 0$.
Using the Mean Value Theorem and the fact that $|g'(x)| \le k$, for any $n \ge 0$, we have
$|p_n - p| = |g(p_{n-1}) - g(p)| = |g'(z_n)||p_{n-1} - p| \le k|p_{n-1} - p|$ for some $z_n \in (a,b)$.
Then $|p_n - p| \le k|p_{n-1} - p| \le k^2|p_{n-2} - p| \le \cdots \le k^n|p_0 - p|$.
Notice that since $|k| < 1$, then $\lim_{n\to\infty} k^n = 0$.

Hence $0 \leq \lim_{n\to\infty} |p_n - p| \leq \lim_{n\to\infty} k^n |p_0 - p_1| = 0$

Thus $\lim_{n\to\infty} |p_n - p| = 0$, and therefore $p_n \to p$.

5. (8 points) Determine the rate of convergence for the limit $\lim_{n\to\infty} \sin\left(\frac{1}{n^k}\right) = 0$.

Let $f(x) = \sin x$. Then $f'(x) = \cos x$. Notice that $f(x)$ is continuous and differentiable on $[0, \frac{1}{n^k}]$ for any $n$ and any $k$.

Therefore, by the Mean Value Theorem, there is a $x \in \left(0, \frac{1}{n^k}\right)$ such that
$\sin\left(\frac{1}{n^k}\right) - \sin(0) \leq \cos z \left(\frac{1}{n^k} - 0\right)$.
Thus $|\sin\left(\frac{1}{n^k}\right) - 0| \leq 1 \cdot \frac{1}{n^k}$ [since $|\cos z| \leq 1$].
Hence this limit has order of convergence $O\left(\frac{1}{n^k}\right)$