Math 450
Exam 2

Instructions: You will have 75 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.
Work your exam on separate sheets of paper. Be sure to put your name on at least the front page.

1. Consider the sequence $\{p_n\}_{n=0}^{\infty}$ where $p_n = \frac{1}{2^n}$

   (a) (3 points) Find $\lim_{n\to\infty} p_n$

   Notice that since 2^n is increasing and 1 is constant, then $\lim_{n\to\infty} \frac{1}{2^n} = 0$.

   (b) (7 points) Prove that this sequence converges linearly.

   By definition, the sequence $\{p_n\}_{n=0}^{\infty}$ converges to p linearly if there is a $\lambda > 0$ such that $\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^1} = \lambda$.

   Notice that $\lim_{n\to\infty} \frac{\left|\frac{1}{2^{n+1}} - 0\right|}{\left|\frac{1}{2^n} - 0\right|} = \lim_{n\to\infty} \frac{2^n}{2^{n+1}} = \frac{1}{2}$.

   Therefore, $p_n \to p$ linearly.

2. (8 points) Let $f(x) = e^x - \frac{1}{6}x^3 - \frac{1}{2}x^2 - x - 1$. Notice that $f(0) = 0$. Determine the multiplicity of the root $x = 0$.

   Recall that according to Theorem 2.11 in your textbook, a function $f \in C^m[a,b]$ has a root of multiplicity m at p in (a,b) if and only if $0 = f(p) = f'(p) = \cdots = f^{(m-1)}(p)$, but $f^{(m)}(p) \neq 0$.

   Now, $f(0) = 0$, $f'(x) = e^x - \frac{1}{2}x^2 - x - 1$, so $f'(0) = 0$, $f''(x) = e^x - x - 1$, so $f''(0) = 0$, $f^{(3)}(x) = e^x - 1$, so $f^{(3)}(0) = 0$, $f^{(4)}(x) = e^x$, so $f'(0) = 1 \neq 0$

   Therefore, 0 is a root of $f(x)$ with multiplicity 4.

3. (10 points) Use Steffensen's Method to approximate a solution to $x = \frac{1}{3}e^x$ by starting with $p_0^{(0)} = 1$ and computing $p_1^{(0)}$, $p_2^{(0)}$, and $p_0^{(1)}$.

   Let $g(x) = \frac{1}{3}e^x$ and let $p_0^{(0)} = 1$.
   Then $p_0^{(1)} = \frac{1}{3}e^1 \approx 0.90609$ and $p_0^{(2)} \approx \frac{1}{3}e^{0.90609} \approx 0.82488$

   Hence $p_1^{(0)} = p_0^{(0)} - \frac{\left(p_0^{(1)} - p_0^{(0)}\right)^2}{p_0^{(2)} - 2p_0^{(1)} + p_0^{(0)}} \approx 1 - \frac{(0.90609 - 1)^2}{(0.82488) - 2(0.90609) + 1} \approx 0.30558$

4. (a) (7 points) Explain the difference between Aitken's Method and Steffensen's method in your own words.

   Answers vary. You were graded based on how clearly you described each algorithm and how their computations differ. The main idea is that both rely on computing the first and second forward difference based on three values computed using standard fixed point iteration. In Aitken's Method, one continues computing values by applying fixed point iteration based on the initial guess p_0, while in Steffensen's Method, once an approximation of the root is obtained from the initial 3 values obtained using fixed point iteration, the algorithm is restarted, using this new approximation as an initial guess and computing 2 new values via fixed point iteration.

(b) (7 points) Recall that given an equation $f(x) = 0$, Newton's Method approximates a solution to the equation by starting with an initial guess p_0 and generating a sequence using the fixed point recursive function $p_{n+1} = g(p_n) = p_n - \frac{f(p_n)}{f'(p_n)}$. Give a convincing argument either for or against the statement: "Applying Steffensen's method to the fixed point function $p_{n+1} = g(p_n)$ gives a sequence that converges to a root p of $f(x)$ more rapidly than the original sequence arising from Newton's method."

The answer to this question depends on what assumptions you made. Credit was given based on how clearly your argument either for or against the statement was stated and based on the accuracy of the reasoning provided.

5. Let $f(x) = x^4 - 4x^3 + 7x^2 - 5x + 2$

(a) (6 points) Use synthetic division to find $f(2)$.

$$\begin{array}{c|ccccc} & 1 & -4 & 7 & -5 & 2 \\ 2 & & 2 & -4 & 6 & 2 \\ \hline & 1 & -2 & 3 & 1 & 4 \end{array}$$

Thus $f(2) = 4$

(b) (6 points) Use synthetic division to find $f'(2)$.

$$\begin{array}{c|cccc} & 1 & -2 & 3 & 1 \\ 2 & & 2 & 0 & 6 \\ \hline & 1 & 0 & 3 & 7 \end{array}$$

Thus $f'(2) = 7$.

6. Given the following data values:

x	0	2	3
$f(x)$	1	5	8

(a) (9 points) Construct the Lagrange Polynomial interpolating $f(x)$ at the given points. [You do not need to simplify.]

$$P_2(x) = (1)\frac{(x-2)(x-3)}{(0-2)(0-3)} + (5)\frac{(x-0)(x-3)}{(2-0)(2-3)} + (8)\frac{(x-0)(x-2)}{(3-0)(3-2)} = \frac{1}{6}(x-2)(x-3) + \frac{5}{-2}x(x-3) + \frac{8}{3}x(x-2)$$

(b) (3 points) Use the Langrange Polynomial you found to approximate $f(1)$.

Using the polynomial we computed above, $P_2(1) = \frac{1}{6}(1-2)(1-3) + \frac{5}{-2}(1)(1-3) + \frac{8}{3}(1)(1-2) = \frac{1}{3} + 5 - \frac{8}{3} = \frac{8}{3} \approx 2.6667$.

7. Given the following equally spaced data values:

x	0	0.2	0.4	0.6
$f(x)$	1	3	8	10

(a) (9 points) Use the Newton Forward Difference Method to find a 3rd degree interpolating polynomial $P_3(s)$, where $s = \frac{x-x_0}{h}$. [You do not need to simplify.]

We begin by constructing the following forward difference table (this table is not essential, but it helps organize the information we need to construct $P(s)$.)

x	f(x)			
0	1			
		2		
0.2	3		3	
		5		−6
0.4	8		−3	
		2		
0.6	10			

Then $P(s) = 1 + \binom{s}{1}(2) + \binom{s}{2}(3) + \binom{s}{3}(-6)$

$= 1 + \dfrac{(s)}{1}(2) = \dfrac{s(s-1)}{2!}(3) + \dfrac{s(s-1)(s-2)}{3!}(-6)$

(b) (3 points) Use the Polynomial $P_3(s)$ you found to approximate $f(0.5)$.

Notice that $s = \frac{x-x_0}{h} = \frac{0.5-0}{0.2} = 2.5$

Therefore, using the polynomial we constructed above, $P(2.5) = 1 + (2.5)(2) + \frac{2.5(1.5)}{2}(3) + \frac{2.5(1.5)(0.5)}{6}(-6) = 1 + 5 + 5.625 - 1.875 = 9.75$.

8. (8 points) Let $P_2(x)$ be the Lagrange interpolating polynomial that interpolates $f(x) = 3x^4 - 2x^3 + 5x^2 - 7x + 1$ using the values of f when $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$. **Without** actually finding $P_2(x)$, find an upper bound on the absolute error in using $P_2(1.5)$ to approximate $f(1.5)$.

First notice that since we have three interpolating values, the related Lagrange polynomial will be of degree 2. Recall that in this case, $E(x) = \dfrac{f^{(3)}(z)}{3!}(x-x_0)(x-x_1)(x-x_3)$ for some $z \in (a,b)$.
Here, $(a,b) = (0,2)$, $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$, and since $f(x) = 3x^4 - 2x^3 + 5x^2 - 7x + 1$,

$f'(x) = 12x^3 - 6x^2 + 10x - 7$, $f''(x) = 36x^2 - 12x + 10$, and $f^{(3)}(x) = 72x - 12$.
We do not know the value of z, but since we wish to find an upper bound of the magnitude of the absolute error, we can compute an upper bound by maximizing $f^{(3)}(x)$ on $[0,2]$. Since $f^{(3)}(x)$ is continuous and increasing, the maximum value occurs when $x = 2$ and the minimum value occurs when $x = 0$.
Since we want the maximum magnitude, we check both values:
$f^{(3)}(0) = 72(0) - 12 = -12$, and $f^{(3)}(2) = 72(2) - 12 = 132$
Thus $\left|f^{(3)}(z)\right| \leq 132$ on $[0,2]$.
Hence $|E(x)| \leq \frac{132}{3!}x(x-1)(x-2)$. Thus when $x = 1.5$, the error is at most $|E(1.5)| = \left|\frac{132}{6}(1.5)(0.5)(-0.5)\right| = 8.25$

9. (8 points) Given that $f(0) = 3$, $f'(0) = 8$, $f(2) = 11$, $f'(2) = 6$, and $f''(2) = -4$, find a 4th degree polynomial interpolating f relative to this data.

We begin with a divided difference table, this time, employing one with repeated values (determined by the values of the first and second derivative that we know). Recall that given interpolation values $x_0 \leq x_1 \leq \cdot \leq x_n$, if $x_0 = x_k$, then $f[x_0, x_1, \cdots, x_k] = \dfrac{f^{(k)}(x_0)}{k!}$)

x	f(x)				
0	3				
		8			
0	3		−2		
		4		3/2	
2	11		1		−3
		6		−3/2	
2	11		−2		
		6			
2	11				

Using this table, we read off the following polynomial:
$P_4(x) = 3 + 8x - 2x^2 + \frac{3}{2}x^2(x-2) - 3x^2(x-2)^2$

Extra Credit: (8 points) Prove the following:
For each natural number n, if $x_0, x_1, \cdots x_n$ are equally spaced numbers (that is, $x_j = x_0 + jh$), then $f[x_0, x_1, \cdots, x_n] = \dfrac{\Delta^n f_0}{n! h^n}$

This is a standard proof using induction. (State and prove the base case, then prove that if case k is true, then case $k+1$ is also true.
[See course notes.]