Math 450
Exam 3 - Solutions

Instructions: You will have 75 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.
Work your exam on separate sheets of paper. Be sure to put your name on at least the front page.

1. (12 points) A clamped cubic spline S for a function f is defined on $[1, 3]$ by:

$$S(x) = \begin{cases} S_0(x) = 3(x-1) + 2(x-1)^2 - (x-1)^3 & \text{if } 1 \le x < 2, \\ S_1(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3 & \text{if } 2 \le x \le 3 \end{cases}$$

Given that $f'(1) = f'(3)$, find a, b, c, and d.

Recall that in order for the piecewise defined spline to be continuous, we must have $S_0(2) = S_1(2)$.
Therefore, $S_0(2) = 3(1) + 2(1)^2 - (1)^3 = 3 + 2 - 1 = 4 = S_1(2) = a + b(0) + c(0)^2 - d(0)^3 = a$, so $a = 4$.
Next, notice that $S_0'(x) = 3 + 4(x-1) - 3(x-1)^2$ and $S_1'(x) = b + 2c(x-2) + 3d(x-2)^2$
Recall that the first derivatives of our pieces must also agree at the point of overlap.
Therefore, $S_0'(2) = 3 + 4 - 3 = 4 = S_1'(2) = b + 2c(0) + 3d(0)^2 = b$, thus $b = 4$.
Continuing, $S_0''(x) = 4 - 6(x-1)$ and $S_1'(x) = 2c + 6d(x-2)$
Recall that the second derivatives of our pieces must also agree at the point of overlap.
Therefore, $S_0''(2) = 4 - 6 = -2 = S_1''(2) = 2c(0) + 6d(0) = 2c$, thus $2c = -2$, or $c = -1$.
Finally, $f'(1) = S_0'(1) = 3 + 4(0) - 3(0)^2 = 3 = f'(3) = S_1'(3) = b + 2c + 3d$. Therefore, $3 = d + 2c + 3d$, or $3 = 4 + 2(-1) + 3d$.
Hence $1 = 3d$, or $d = \frac{1}{3}$.

2. Let $f(x) = \sin x$ and assume that we **only** have access to the specific values given in the table below:

x	0.80	1.20
$\sin x$	0.717356	0.932039

(a) (6 points) Use the most accurate method (among those discussed in class) to approximate $f'(1)$

The most accurate method available here is the three point "middle" approximation. Recall that this approximation has the form:
$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} - \frac{h^2}{6} f^{(3)}(z_1)$. Here, $x_0 = 1$, and $h = 0.2$.

Using this approximation, $f'(1) \approx \frac{f(1.2) - f(0.8)}{2(0.2)} \approx \frac{0.932039 - 0.717356}{0.4} = 0.5367075$.

(b) (8 points) Find the error term for the approximation method you used to find an upper bound for the error in the approximations you found in part (a).

The error term is given by: $\frac{h^2}{6} f^{(3)}(z_1)$. Since $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, and so $f^{(3)}(x) = -\cos x$. Notice that since $\cos x$ is decreasing on the interval $[0, \pi]$, the maximum value of $|-\cos x|$ on the interval $[0.8, 1.2]$ will occur when $x = 0.8$. Therefore, an upper bound on the error is: $E \le \frac{(0.2)^2}{6}(\cos 0.8) \approx 0.004644711$.

3. (15 points) Assume that $P(x)$ interpolates $f(x)$ at x_0 and x_1 where $x_1 = x_0 + h$.
Then $f(x) = \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1) + \frac{f''(z(x))}{2!}(x-x_0)(x-x_1)$.
Differentiate this expression for $f(x)$, then evaluate it at x_0.
By doing so, you will obtain a formula for approximating $f'(x_0)$ along with the error term for this approximation.

Differentiating $f(x)$ [making sure to use the product and chain rule on the last term] yields:
$f'(x) = \frac{f(x_0)}{x_0-x_1} + \frac{f(x_1)}{x_1-x_0} + (2x - x_0 - x_1)\frac{f''(z(x))}{2} + (x-x_0)(x-x_1)\frac{f'''(z(x))}{2} z'(x)$.
Evaluating at $x = x_0$ and simplifying a bit using the fact that $x_1 = x_0 + h$ gives:
$\frac{f(x_0)}{-h} + \frac{f(x_0+h)}{h} + (2x_0 - x_0 - (x_0+h))\frac{f''(z(x_0))}{2} + (x_0-x_0)(x_0-x_1)\frac{f'''(z(x_0))}{2} z'(x_0)$
$= \frac{f(x_0)}{-h} + \frac{f(x_0+h)}{h} + (-h)\frac{f''(z(x_0))}{2} + (0)(-h)\frac{f'''(z(x_0))}{2} z'(x_0)$
$= \frac{1}{h}\left(f(x_0+h) - f(x_0)\right) - \frac{h}{2} f''(z(x_0))$

4. (15 points) Suppose $N(h)$ is an approximation of a quantity M for every $h > 0$ and that
$M = N(h) + K_1 h^2 + K_2 h^4 + K_3 h^6 + \cdots$.
Prove that $M = N_2(h) = \frac{9N(\frac{h}{3}) - N(h)}{8}$ is a $O(h^4)$ approximation of M.

Notice that since $M = N(h) + K_1 h^2 + K_2 h^4 + K_3 h^6 + \cdots$, then $M = N(\frac{h}{3}) + K_1\left(\frac{h}{3}\right)^2 + K_2\left(\frac{h}{3}\right)^4 + K_3\left(\frac{h}{3}\right)^6 + \cdots$
or $M = N(\frac{h}{3}) + K_1 \frac{h^2}{9} + K_2 \frac{h^4}{81} + K_3 \frac{h^6}{729} + \cdots$.
Hence $9M = 9N(\frac{h}{3}) + K_1 h^2 + K_2\left(\frac{h^4}{9}\right) + K_3\left(\frac{h^6}{81}\right) + \cdots$.
While $M = N(h) + K_1 h^2 + K_2 h^4 + K_3 h^6 + \cdots$

Subtracting these, $8M = 9N(\frac{h}{3}) - N(h) + K_2\left(\frac{h^4}{9} - h^4\right) + K_3\left(\frac{h^6}{81} - h^6\right) + \cdots$

Thus $M = \frac{9}{8}N(\frac{h}{3}) - \frac{1}{8}N(h) + \frac{1}{8}K_2\left(-\frac{8h^4}{9}\right) + \frac{1}{8}k_3\left(-\frac{80h^6}{81}\right) + \cdots$

Therefore $M = \frac{9N(\frac{h}{3}) - N(h)}{8} - \frac{1}{9}K_2 h^4 - \frac{10}{81}k_3 h^6 + \cdots$
Which establishes that $M = \frac{9N(\frac{h}{3}) - N(h)}{8}$ is a $O(h^4)$ approximation of M.

5. Let $f(x) = e^{x^2}$

(a) (6 points) Use the Trapezoid Rule to approximate $\int_0^1 f(x)\,dx$.

$$\int_0^1 f(x)\,dx \approx \frac{1}{2}\left(f(1) + f(0)\right) = \frac{e+1}{2} \approx 1.859140914.$$

(b) (6 points) Find an an upper bound on the error in your approximation using the error term from the Trapezoid Rule formula.

Recall that the error term for the Trapezoid Rule is given by $E \le |\frac{h^3}{12} f''(z)|$.
Here, $f'(x) = 2xe^{x^2}$, and $f''(x) = 2e^{x^2} + 4x^2 e^{x^2}$. Notice that $f''(x)$ is positive and increasing for $x \ge 0$, so the maximum value of $|f''(z)|$ on $[0,1]$ will occur when $x = 1$. $|f''(1)| = 2e^1 + 4(1)^2 e^{1^2} = 2e + 4e = 6e$. Also, $h = 1$
Hence the Error is bounded above by: $E \le \frac{1^3}{12}(6e) \approx 1.359140914$.

6. Let $g(x) = \ln x$.

(a) (8 points) Use Composite Simpson's Rule with $n = 4$ to approximate $\int_1^3 g(x)\,dx$.

Notice that $h = \frac{3-1}{4} = 0.5$ and we are performing Composite Simpson's Rule on $[1,3]$.
Therefore, $\int_1^3 g(x)\,dx \approx \frac{0.5}{3}\left[\ln 1 + 4\ln\frac{3}{2} + 2\ln 2 + 4\ln\frac{5}{2} + \ln 3\right] \approx 1.295321668.$

(b) (10 points) Determine the values of n and h necessary to approximate $\int_1^3 g(x)\,dx$ to within 10^{-3} using Composite Simpson's Rule.

Recall that the error term for Composite Simpson's Rule is given by: $E \leq |\frac{b-a}{180}h^4 f^{(4)}(z)|$.
We want to find h and n so that $E \leq 0.001$. We first compute the fourth derivative of $f(x)$:
$f'(x) = \frac{1}{x}$, $f''(x) = \frac{-1}{x^2}$, $f^{(3)}(x) = \frac{2}{x^3}$, and $f^{(4)}(x) = \frac{-6}{x^4}$. Notice that $f^{(4)(x)}$ takes on its maximum value on $[1, 3]$ when $x = 1$, yielding a value of $f^{(4)}(1) = -6$.

Now, If $E \leq |\frac{b-a}{180}h^4 f^{(4)}(z)| = \frac{2}{180}h^4 f^{(4)}(z) \leq \frac{6h^4}{90}$.
If $\frac{6h^4}{90} = 0.001$, then $h^4 = 0.015$, so $h = (0.015)^{\frac{1}{4}} \approx 0.34496$.
Since $n = \frac{b-a}{h} = \frac{2}{h}$, then $n \approx \frac{2}{.3496} \approx 5.71494$. Rounding up, we take $n = 6$, and $h = \frac{1}{3}$.

7. (15 points) Use Romberg Integration and Richardson Extrapolation to approximate $\int_0^1 f(x)\,dx$ as accurately as possible using *only* the data in the following table:

x	0	0.25	0.50	0.75	1.0
$f(x)$	0	0.062581508	0.255341921	0.630437674	1.557407725

Recall that when using Romberg integration, $R_{1,1} = \frac{h_1}{2}[f(a) + f(b)]$, and then $R_{k,1}$ is defined recursively as:

$$R_{k,1} = \frac{1}{2}\left[R_{k-1,1} + h_{k-1}\sum_{i=1}^{2^{k-2}} f(a + (2i-1)h_k)\right].$$

From this, notice that here, $h_1 = 1$ is the largest increment represented by our table of values, and we can continue two more steps by taking $h_2 = \frac{1}{2}$, and $h_3 = \frac{1}{4}$.

Then $R_{1,1} = \frac{h_1}{2}[f(a) + f(b)] = \frac{1}{2}(0 + 1.557407725) \approx 0.778703863$
$R_{2,1} = \frac{1}{2}[R_{1,1} + f(0.5)] = \frac{1}{2}(0.778703863 + 0.255341921) \approx 0.517022892$
$R_{3,1} = \frac{1}{2}[R_{2,1} + \frac{1}{2}f(0.25) + \frac{1}{2}f(.75)] \approx 0.43176624$

From here, we use Richardson's Extrapolation to combine our base approximations which we computed above using the Composite Trapezoid Rule for different h values. The Extrapolation Recursion is given by: $R_{k,j} = R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^{j-1} - 1}$ or $R_{k,j} = \frac{4R_{k,j-1} - R_{k-1,j-1}}{4^{j-1} - 1}$.
Then $R_{2,2} = \frac{4R_{2,1} - R_{1,1}}{3} \approx 0.429795902$.
$R_{3,2} = \frac{4R_{3,1} - R_{2,1}}{3} \approx 0.403347357$.
$R_{3,3} = \frac{16R_{3,2} - R_{2,2}}{15} \approx 0.401584121$.