Math 450
Exam 4

Instructions: You will have 75 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.
Work your exam on separate sheets of paper. Be sure to put your name on at least the front page.

1. (a) (8 points) Use Gaussian Quadrature with $n = 3$ to approximate the integral $\int_1^2 x^3 \ln x \, dx$.

   Recall that when using Gaussian Quadrature, $\int_{-1}^1 f(x)\, dx \approx \sum_{i=1}^n c_i f(r_i)$, where $c_i = \int_{-1}^1 \prod_{j=1, j\neq i}^n \frac{x - x_j}{x_i - x_j}\, dx$ and r_i is a root of the Legendre polynomial of degree n.

   First, we transform the integral so that it is a definite integral on $[-1, 1]$:
   $\int_1^2 x^3 \ln x \, dx = \int_{-1}^1 \left(\frac{t+3}{2}\right)^3 \ln\left(\frac{t+3}{2}\right) \cdot \frac{1}{2}\, dt$

   Next, we approximate the value of the transformed integral by using the values of c_i and r_i when $n = 3$ (see the table of legendre roots and coeffeicients).
   Thus $\int_1^2 x^3 \ln x \, dx = \int_{-1}^1 \left(\frac{t+3}{2}\right)^3 \ln\left(\frac{t+3}{2}\right) \cdot \frac{1}{2}\, dt \approx \sum_{i=1}^n c_i \cdot \left(\frac{r_i+3}{2}\right)^3 \ln\left(\frac{r_i+3}{2}\right) \cdot \frac{1}{2} \approx 1.835086883$

   (b) (8 points) Compare your approximation to the actual answer. What is the absolute error of your approximation?

   To find the exact solution, we apply integration by parts. Let $u = \ln x$ and $dv = x^3\, dx$. Then $du = \frac{1}{x}\, dx$ and $v = \frac{1}{4}x^4$.

   Therefore, $\int_1^2 x^3 \ln x \, dx = uv\Big|_1^2 - \int_1^2 v\, du = \frac{1}{4}x^4 \ln x\Big|_1^2 - \frac{1}{4}\int_1^2 x^3\, dx$.
   $= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4\Big|_1^2 = (4\ln 2 - 1) - \left(0 - \frac{1}{16}\right) \approx 1.835088722.$

   Hence, the absolute error of our previous approximation is $|4\ln 2 - \frac{15}{16} - 1.835086883| \approx 0.000001839$

2. Given the initial value problem $y' = \frac{1+t}{1+y}$, $1 \le t \le 2$, $y(1) = 2$

   (a) (10 points) Show that this initial value problem is well posed.

   Some of you were concerned about the fact that $f(t, y)$ is not continuous when $y = -1$. However, a glance at the actual solution to this IVP as given in part $2f$ shows that -1 is not a possible output for the solution function. We will show that this IVP is well posed by showing it is Lipscitz in y on the implied domain of the given expression.

   Notice that $\left|\frac{\partial f}{\partial y}\right| = \left|(1+t)(-1)(1+y)^{-2}\right| = \left|\frac{1+t}{(1+y)^2}\right| \le |1+t| \le 3$ (since $(1+y)^2 \ge 1$, and $1 \le t \le 2$). Hence $y' = f(t, y)$ is Lipschitz in y with constant $L = 3$.

   (b) (10 points) Use Euler's Method to approximate $y(2)$ using $h = 0.5$.

   Recall that in Euler's Method, we take $w_0 = y_0$, and $w_{i+1} = w_i + h \cdot f(t_i, w_i)$.
   Here, $w_0 = 2$, $w_1 = 2(0.5)\left(\frac{1+1}{1+2}\right) = \frac{7}{3}$.
   $y(2) \approx w_2 = \frac{7}{3} + (0.5)\left(\frac{1+1.5}{1+\frac{7}{3}}\right) \approx 2.708333333.$

(c) (10 points) Use the result of theorem 5.9 to find an upper bound in the error in the approximation for found in part (b). Recall that $|y(t_i) - w_i| \leq \frac{hM}{2L}\left[e^{L(t_i-a)} - 1\right]$.

First recall from (a) that $L = 3$ is a Lipschitz constant for $f(t, y)$ for $1 \leq t \leq 2$ and y in the implied domain of this initial value problem.

Next, we use implicit differentiation to compute

$$y'' = \left(\frac{1}{1+y}\right) + (1+t)(-1)\left(\frac{1}{(1+y)^2}\right)(y') = \left(\frac{1}{1+y}\right) + (1+t)\left(\frac{-1}{(1+y)^2}\right)\left(\frac{1+t}{1+y}\right) = \frac{(y+1)^2 - (1+t)^2}{(y+1)^3}$$

Note that since $y(t) = \sqrt{t^2 + 2t + 6} - 1$, is an increasing function on $1 \leq t \leq 2$, then $2 \leq y \leq \sqrt{14} - 1$

Hence $|y''| \leq \frac{(\sqrt{14}-1)^2 - 2^2}{3^2} = \frac{14-4}{27} = \frac{10}{27} \approx 0.37037037 = M$

Amending the L found above, if we take $y = 2$ as the minimum value of y,

$\left|\frac{\partial f}{\partial y}\right| = \left|(1+t)(-1)(1+y)^{-2}\right| = \left|\frac{1+t}{(1+y)^2}\right| \leq |\frac{1+2}{9}| \leq \frac{1}{3} = L$ will give us a better bound.

Using these, $|y(2) - w_2| \leq \frac{(0.5)(\frac{10}{27})}{2(\frac{1}{3})}\left[e^{\frac{1}{3}(1)} - 1\right] \approx 0.1098923403$.

(d) (10 points) Use Taylor's Method of order 2 to approximate $y(2)$ using $h = 0.5$.

Recall that in Taylor's Method of Order 2, $T^{(2)} = f(t, y) + \frac{h}{2}(f'(t, y)) = \frac{1+t}{1+y} + \frac{h}{2}\left(\frac{(y+1)^2 - (1+t)^2}{(y+1)^3}\right)$

See part (b) for the derivative computation

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Also, $w_0 = y_0$, and $w_{i+1} = w_i + hT^{(2)}$.

Then $w_0 = 2$, $w_1 = 2 + 0.5\left[\frac{1+1}{1+2} + \frac{0.5}{2}\left(\frac{(2+1)^2 - (1+1)^2}{(2+1)^3}\right)\right] \approx 2.356481481$.

Thus $y(2) \approx w_2 = w_1 + 0.5\left[\frac{1+1.5}{1+w_1} + \frac{0.5}{2}\left(\frac{(w_1+1)^2 - (1+1.5)^2}{(w_1+1)^3}\right)\right] \approx 2.745476323$.

(e) (10 points) Use the Runge-Kutta Order Four method to approximate $y(2)$ using $h = 0.5$.

Recall that in Runge-Kutta Order Four, $w_0 = y_0$,

$k_1 = h \cdot f(t_i, y_i)$, $k_2 = h \cdot f(t_i + \frac{h}{2}, y_i + \frac{1}{2}k_1)$, $k_3 = h \cdot f(t_i + \frac{h}{2}, y_i + \frac{1}{2}k_2)$, $k_4 = h \cdot f(t_i + h, y_i + k_3)$, and $w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$.

Then, in this case, $w_0 = 2$, and when computing w_1, we see that $k_1 = \frac{1}{3}$, $k_2 = \frac{27}{76}$, $k_3 = \frac{57}{161}$, and $k_4 = \frac{161}{432}$, so $w_1 = 2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \approx 2.354103229$.

Continuing, when computing w_2, we see that $k_1 = 0.3726778560$, $k_2 = 0.3883695706$, $k_3 = 0.3875108206$, and $k_4 = 0.4008965062$, so $w_2 = w_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \approx 2.741659086$.

(f) (10 points) Prove that $y(t) = \sqrt{t^2 + 2t + 6} - 1$ is a solution to this initial value problem.

Notice that $y(1) = \sqrt{(1)^2 + 2(1) + 6} - 1 = \sqrt{9} - 1 = 3 - 1 = 2$.

Also, $y'(t) = \frac{1}{2}(t^2 + 2t + 6)^{\frac{1}{2}} \cdot (2t + 2) = \frac{t+1}{\sqrt{t^2+2t+6}} = \frac{t+1}{y+1}$.

Hence, this is a solution to the original initial value problem.

(g) (10 points) Use $y(t) = \sqrt{t^2 + 2t + 6} - 1$ to compute the actual value of $y(2)$. Then find the absolute error of your previous approximations.

Notice that using $y(t) = \sqrt{t^2 + 2t + 6} - 1$, $y(2) = \sqrt{4 + 4 + 6} - 1 = \sqrt{14} - 1 \approx 2.741657387$.

Comparing this to the result from (b), the absolute error is: $|(\sqrt{14} - 1) - 2.708333333| \approx 0.033324054$.

Comparing this to the result from (d), the absolute error is: $|(\sqrt{14} - 1) - 2.745476323| \approx 0.003818936$.

Comparing this to the result from (e), the absolute error is: $|(\sqrt{14} - 1) - 2.741659086| \approx 0.000001699$.

3. Prove **one** of the following. If you attempt more than one, you must make it clear which problem you want to have graded.

   (a) (15 points) **Theorem 5.3** Suppose $f(t, y)$ is defined on a convex set $D \subseteq \mathbb{R}^2$. If a constant $L > 0$ exists with $\left|\frac{\partial f}{\partial y}(t, y)\right| \leq L$, for all $(t, y) \in D$, then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L.

   See course notes.

   (b) (20 points) **Theorem 4.7** Suppose that $x_1, x_2, \cdots, x_n$ are the roots of the nth degree Legendre Polynomial $P_n(x)$ and that, for each $i = 1, 2, \cdots, n$, the numbers c_i are defined by: $c_i = \int_{-1}^{1} \prod_{j=1, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}\, dx$. If $P(x)$ is any polynomial of degree less than $2n$, then $\int_{-1}^{1} P(x)\, dx = \sum_{i=1}^{n} c_i P(x_i)$. [Hint: First prove the case where deg $P < n$.]

   See course notes.