Math 310
Exam 2 - Practice Problems

1. Prove that there is an integer m such that $m^3 > 10^{10}$. Is your proof constructive or non-constructive?
2. Prove that given any two rational numbers $p < q$, there is a rational number r with $p < r < q$.
3. Prove that given a non-negative integer n, there is a unique non-negative integer m such that $m^2 \le n < (m+1)^2$
4. Prove or disprove: Every non-negative integer can be written as the sum of at most 3 perfect squares.
5. Formulate a conjecture about the final two digits of the square of any integer. Then prove your conjecture.
6. For each of the following, determine whether the statement is True or False.
   (a) $\emptyset \subseteq \{a, b, c, d\}$
   (b) $\emptyset \in \{a, b, c, d\}$
   (c) $\emptyset \in \{a, b, \emptyset\}$
   (d) $\emptyset \subseteq \{a, b, \emptyset\}$
   (e) $\{a, b\} \subset \{a, b\}$
   (f) $0 \in \{0, \{1\}, \{0, 1\}\}$
   (g) $1 \in \{0, \{1\}, \{0, 1\}\}$
   (h) $\{0, 1\} \in \{0, \{1\}, \{0, 1\}\}$
   (i) $\{0, 1\} \subset \{0, \{1\}, \{0, 1\}\}$
7. Given the set $B = \{a, b, \{a, b\}\}$
   (a) Find $|B|$.
   (b) Find $\mathcal{P}(B)$
8. Given that $A = \{1, 2, 3\}$ and $B = \{a, b, c, d, e, f\}$
   (a) List the elements in $A \times A$.
   (b) How many elements are in $A \times B$?
   (c) How many elements are in $A \times (B \times B)$?
9. Find the set of all elements that make the predicate $Q(x) : x^2 < x$ true (where the domain of x is all real numbers).
10. Given that $A = \{0, 2, 4, 6, 8, 10, 12\}$, $B = \{0, 2, 3, 5, 7, 11, 12\}$ and $C = \{1, 2, 3, 4, 6, 7, 8, 9\}$ are all subsets of the universal set $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, find each of the following:
    (a) $A - B$
    (b) $\overline{A}$
    (c) $A \cap B$
    (d) $A \cup (B \cap C)$
    (e) $A - (\overline{B} \oplus C)$
    (f) $(A \cap C) \cup (B - \overline{A})$
11. Draw Venn Diagrams representing each of the following sets:
    (a) $A - B$
    (b) $B - \overline{A}$
    (c) $(A \cup C) \cap B$
    (d) $\overline{A \cup B \cup C}$
    (e) $A - (B \cup C)$
    (f) $(A \cap B) - \overline{C}$
12. Use a membership table to show that $(B - A) \cup (C - A) = (B \cup C) - A$.
13. Use a 2-column proof to verify the set identity: $A \cup (A \cap B) = A$.
14. Use a paragraph (double containment) proof to show that $A - B = A \cap \overline{B}$.
15. For each of the following, either prove the statement or show that it is false using a counterexample.
    (a) $(A - B) - C = A - (B - C)$
    (b) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$
    (c) $A \cap (B - C) = (A \cap B) - (A \cap C)$
16. Consider the function $f(x) = |x|$
    (a) Suppose that the domain of this function is $\mathbb{R}$ and the co-domain is $\mathbb{R}$. Find the range of f. Is f 1-1? Is f onto? Justify your answers.
    (b) Suppose that the domain of this function is $\mathbb{N}$ and the co-domain is $\mathbb{N}$. Find the range of f. Is f 1-1? Is f onto? Justify your answers.
    (c) Suppose $S = \{-2, -1, 0, 1, 2\}$. Find $f(S)$ (the image of the set S under f). Find $f^{-1}(S)$ (the preimage of the set S under f).

17. For each of the following functions, determine whether f is a one-to-one. Also determine whether f is onto. Justify your answers.

   (a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - x$
   (b) $f : \mathbb{R}^+ \to \mathbb{R}^+$ $f(x) = x^2$
   (c) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ $f(m, n) = m^2 - n$
   (d) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ $f(m, n) = m^2 - n^2$

18. Graph the following functions. Assume that the domain is $\mathbb{R}$.

   (a) $f(x) = \lceil x \rceil - 1$
   (b) $g(x) = \lfloor \frac{x-1}{2} \rfloor$
   (c) $h(x) = \lceil x \rceil - \lfloor x + 1 \rfloor$

19. Prove or Disprove: Suppose $f : B \to C$ and $g : A \to B$. If f is one-to-one and g is onto, then $f \circ g$ is one to one.

20. Prove or Disprove: Suppose $f : B \to C$ and $g : A \to B$. If f is one-to-one and g is onto, then $f \circ g$ is onto.

21. Prove that $n^5 - n$ is divisible by 5 for any non-negative integer n.

22. Prove that for $r \in \mathbb{R}$, $r \neq 1$ and for all integers n, $\displaystyle\sum_{j=0}^{n} r^j = \frac{r^{n+1} - 1}{r - 1}$

23. Prove that for all $n \geq 2$, $\displaystyle\sum_{k=1}^{n} \frac{1}{k^2} < 2 - \frac{1}{n}$

24. Prove that $n! < n^n$ whenever $n > 1$.

25. Prove that for all n, $\displaystyle\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$

26. Suppose that $f(x) = e^x$ and $g(x) = xe^x$. Use induction and the product rule to show that $g^{(n)}(x) = (x + n)e^x$ for all $n \geq 1$.

27. Given the relation $R = \{(1, 1), (1, 3), (1, 4), (2, 2), (3, 1), (3, 4), (4, 1), (4, 3)\}$ on the set $A = \{1, 2, 3, 4\}$:

   (a) Determine whether or not R is reflexive.
   (b) Determine whether or not R is irreflexive.
   (c) Determine whether or not R is symmetric.
   (d) Determine whether or not R is antisymmetric.
   (e) Determine whether or not R is transitive.

28. Given the relation $S = \{(1, 1), (1, 3), (2, 1), (2, 2), (2, 3), (3, 3), (4, 4)\}$ on the set $A = \{1, 2, 3, 4\}$:

   (a) Determine whether or not S is reflexive.
   (b) Determine whether or not R is irreflexive.
   (c) Determine whether or not R is symmetric.
   (d) Determine whether or not R is antisymmetric.
   (e) Determine whether or not R is transitive.

29. Suppose that R and S are symmetric relations on a non-empty set A. Prove or disprove each of these statements:

   (a) $R \cup S$ is symmetric.
   (b) $R \cap S$ is symmetric.
   (c) $R - S$ is symmetric.
   (d) $R \oplus S$ is symmetric.
   (e) $S \circ R$ is symmetric.