

**Definitions:**

- A relation  $R$  on a set  $S$  is a **partial ordering** or **partial order** if it is reflexive, antisymmetric, and transitive.
- A set  $S$  together with a partial ordering  $R$  is called a **partial ordered set** or **poset**, denoted by  $(S, R)$  or  $(S, \preceq)$ .
- Members of the set  $S$  are called **elements** of the poset.
- The elements  $a$  and  $b$  of a poset  $(S, \preceq)$  are called **comparable** if either  $a \preceq b$  or  $b \preceq a$ . If neither of these occur for a pair  $a$  and  $b$ , then we say that these elements are **incomparable**.
- If  $(S, \preceq)$  is a poset and every pair of elements in  $S$  are comparable, then we say that  $S$  is a **totally ordered set** and we say that  $\preceq$  is a **total order**.
- If  $S$  is a totally ordered set with total ordering  $\preceq$  in which every non-empty set has a least element, then we say that  $(S, \preceq)$  is a **well-ordered set**.

**Examples:**

1.  $(\mathbb{R}, \geq)$  is a poset.
2.  $(\mathbb{Z}, \leq)$  is a poset.
3.  $(\mathbb{Z}, |)$  is a poset.
4. Given a set  $A$ ,  $(\mathcal{P}(A), \subseteq)$  is a poset.
5. Claim:  $(\mathbb{Z}^+, \leq)$  is a well-ordered set.
6. Claim:  $(\mathbb{R}, \leq)$  is not a well-ordered set.
7. Extra Credit: Find an order  $\preceq$  on  $\mathbb{R}$  that makes  $(\mathbb{R}, \preceq)$  a well ordered set.

**Lexicographic Ordering:** Given two posets  $(A_1, \preceq_1)$  and  $(A_2, \preceq_2)$ , the **lexicographic ordering**  $\preceq$  on  $A_1 \times A_2$  is defined as follows:

Given two elements  $(a_1, a_2)$  and  $(b_1, b_2)$ , the pair  $(a_1, a_2) \preceq (b_1, b_2)$  if either  $a_1 \preceq_1 b_1$  or  $a_1 = b_1$  and  $a_2 \preceq_2 b_2$ . Note that  $(a_1, a_2) = (b_1, b_2)$  if  $a_1 = b_1$  and  $a_2 = b_2$ .

**Example:** Consider  $(\mathbb{Z}^+ \times \mathbb{Z}^+, \preceq)$  where we use the standard  $\leq$  ordering on each component.

- (a) Notice that  $(3, 17) \preceq (4, 11)$  since  $3 \leq 4$ .
- (b) Similarly,  $(5, 2) \preceq (5, 7)$  since  $5 = 5$  and  $2 \leq 7$ .

**Note:** We can expand this definition to  $n$ -tuples of elements by applying the same process of comparing the elements in coordinates working from left to right. For example,  $(2, 12, 35) \preceq (3, 1, 4)$ ,  $(3, 2, 17, 11) \preceq (3, 5, 8, 1)$ , and  $(2, 5, 8, 11, 17) \preceq (2, 5, 8, 11, 22)$  in the lexicographic orderings induced by using  $\leq$  in each component. We can get more exotic lexicographic orderings by applying different orderings in one or more of the components.

**More Definitions:**

- An element  $a$  in a poset  $(S, \preceq)$  is **maximal** if there is no element  $b$  such that  $a \prec b$ . Similarly, an element  $a$  in a poset  $(S, \preceq)$  is **minimal** if there is no element  $b$  such that  $b \prec a$ .
- An element  $a$  in a poset  $(S, \preceq)$  is the **greatest element** if  $b \preceq a$  for all  $b \in S$ . Similarly, an element  $a$  is the **least element** if  $a \preceq b$  for all  $b \in S$ .
- Given a poset  $(S, \preceq)$  and a subset  $A \subseteq S$ , an element  $u$  is an **upper bound** for  $A$  if  $a \preceq u$  for all  $a \in A$ . An element  $x$  is the **least upper bound** of  $A$  if  $x$  is an upper bound of  $A$ , and given any upper bound  $u$  of  $A$ ,  $x \preceq u$ .
- Given a poset  $(S, \preceq)$  and a subset  $A \subseteq S$ , an element  $l$  is a **lower bound** for  $A$  if  $l \preceq a$  for all  $a \in A$ . An element  $y$  is the **greatest lower bound** of  $A$  if  $y$  is a lower bound of  $A$ , and given any lower bound  $l$  of  $A$ ,  $l \preceq y$ .
- A poset in which every pair of elements has both a least upper bound and a greatest lower bound is called a **lattice**.

**Example:**