Math 323
Exam 1 - Version 2 Solutions **Name:**______________________________

Instructions: You will have 55 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.

1. Given the curve: $\mathcal{C} = \begin{cases} x = t^4 + 1 \\ y = t^2 - 1 \end{cases}$ for $-1 \le t \le 1$

(a) (8 points) Find an *explicit equation* for the underlying equation in terms of x and y containing the graph of C.

Notice that since $y = t^2 - 1$, so $y + 1 = t^2$.

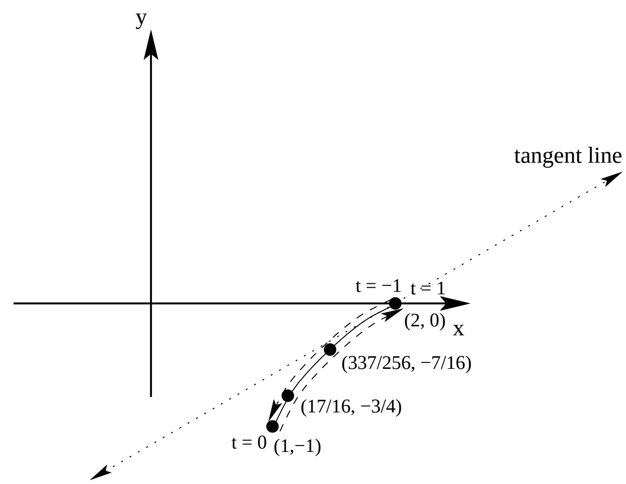
Substituting this into the equation $x = t^4 + 1$ yields $x = (y+1)^2 + 1$, or $x = y^2 + 2y + 1 + 1$

Therefore, an explicit equation for the underlying graph of this parametric curve is: $x = y^2 + 2y + 2$ [I also accepted $y = \sqrt{x-1} - 1$]

(b) (8 points) Graph $\mathcal{C}$, indicating the orientation and labeling at least four points on the graph.

We begin by computing a table of values. Then we plot the related points, labeling both the orientation and the related t-values.

t	x	y
-1	2	0
0	1	-1
1	2	0
$\frac{1}{2}$	$\frac{17}{16}$	$-\frac{3}{4}$
$\frac{3}{4}$	$\frac{337}{256}$	$-\frac{7}{16}$



(c) (8 points) Find the equation of the *tangent line* to $\mathcal{C}$ when $t = -1$ and sketch the associated tangent line on your graph above.

To find the equation for the desired tangent line, we first find the slope of the tangent line.

Recall that $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{2t}{4t^3} = \dfrac{1}{2t^2}$. When $t = -1$, $\dfrac{dy}{dx} = \dfrac{1}{2}$

Next, we notice that the point on the curve then $t = -1$ is $(2, 0)$. We then use the point/slope formula to find an equation for the tangent line:

$y - 0 = \frac{1}{2}(x - 2)$, or $y = \frac{1}{2}x - 1$. See the graph above.

(d) (8 points) Set up (But DO NOT evaluate) an integral with respect to t representing the arc length of C for $0 \le t \le 1$.

Recall that for parametric curves, the arc length differential is: $ds = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\, dt.$

Therefore, $L = \displaystyle\int_0^1 \sqrt{(4t^3)^2 + (2t)^2}\, dt = \int_0^1 \sqrt{16t^6 + 4t^2}\, dt.$

2. (12 points) Find an equation *in rectangular coordinates* for the polar equation: $r = \frac{1}{\sin\theta} + \frac{1}{\cos\theta}$

There are several methods that work, but one of the most straightforward is to begin by dividing both sides by r:

$\dfrac{r}{r} = \dfrac{1}{r\sin\theta} + \dfrac{1}{r\cos\theta}$, so $1 = \frac{1}{y} + \frac{1}{x}$.

Many of you got this far and stopped. However, you should recall that, as discussed in class, if possible, we should find an explicit equation of the form $y = f(x)$. Since it is possible to do so here, only students who completed this simplification received full marks.

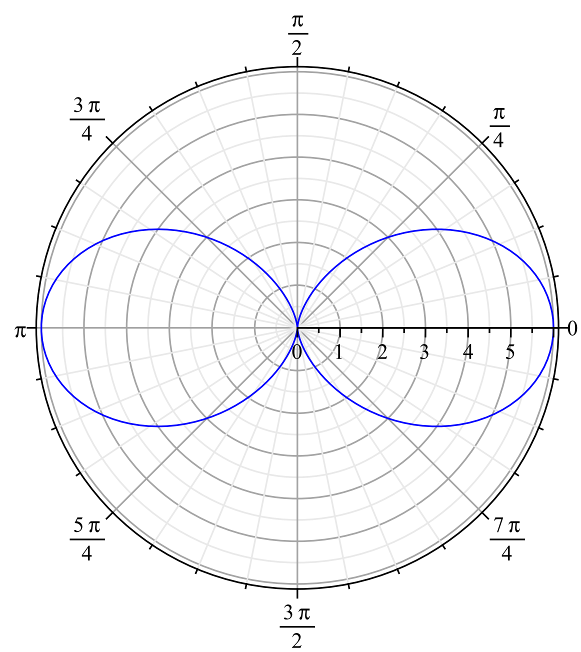
Notice that $1 = \dfrac{x+y}{xy}$, so $xy = x + y$.

Therefore, $xy - y = x$, or $y(x-1) = x$. Therefore, $y = \frac{x}{x-1}$.

3. (12 points) **Graph** the polar equation $r = 3 + 3\cos(2\theta)$. Be sure to indicate the orientation and label *at least four points.*

Since this function is π periodic (as a trigonometric function), I expected you to compute the key values, which are all multiples of $\frac{\pi}{4}$.

θ	2θ	$r = 3 + 3\cos(2\theta)$
0	0	6
$\frac{\pi}{4}$	$\frac{\pi}{2}$	3
$\frac{\pi}{2}$	π	0
$\frac{3\pi}{4}$	$\frac{3\pi}{2}$	3
π	2π	6
$\frac{5\pi}{4}$	$\frac{5\pi}{2}$	3
$\frac{3\pi}{2}$	3π	0
$\frac{7\pi}{4}$	$\frac{7\pi}{2}$	3
2π	4π	6



4. Given the points: $P(-3, 2, -1)$ and $Q(3, -4, 2)$

   (a) (6 points) Find $\overrightarrow{PQ}$

   $\overrightarrow{PQ} = \langle 3 - (-3), -4 - 2, 2 - (-1) \rangle = \langle 6, -6, 3 \rangle$

   (b) (6 points) Find a unit vector in the **opposite** direction as $\overrightarrow{PQ}$.

   First, $\|\overrightarrow{PQ}\| = \sqrt{6^2 + (-6)^2 + (3)^2} = \sqrt{36 + 36 + 9} = \sqrt{81} = 9$

   Then a unit vector in the **opposite** direction as $\overrightarrow{PQ}$ is:

   $$(-1)\frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = \left\langle \frac{-6}{9}, \frac{6}{9}, \frac{-3}{9} \right\rangle = \left\langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle$$

   (c) (6 points) Find an equation for the sphere containing both P and Q. [Hint: What is the center of this sphere?]

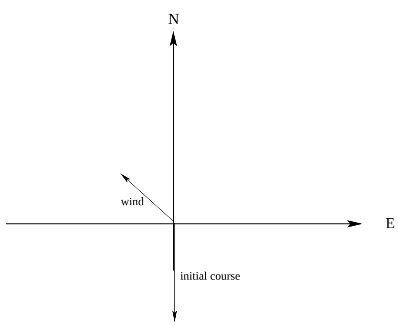
   First notice that the center of this sphere is the midpoint of the segment $\overline{PQ}$, which is given by $\left(\frac{-3+3}{2}, \frac{2-4}{2}, \frac{-1+2}{2} \right) = \left(0, -1, \frac{1}{2} \right)$

   Next, the radius of this sphere is **half** of the distance between P and Q. Notice that we already computed this when we computed $\|\overrightarrow{PQ}\| = 9$. Then $r = \frac{9}{2}$

   Hence an equation for the sphere is: $x^2 + (y+1)^2 + \left(z - \frac{1}{2} \right)^2 = \frac{81}{4}$

5. (10 points) Suppose that the pilot of an airplane sets a course due South at 250 mph. Further suppose that there is a steady wind of 30mph toward the Northwest. Find the resulting speed and direction of the airplane. You should give your answer as a speed in miles per hour along with a heading measured clockwise from due North.

   We begin by drawing a diagram of this situation:

   Notice that we are given information about the wind $\vec{w}$, and the initial course of the airplane $\vec{p}$. We first find the polar representation of these two vectors. Then, we solve for the resultant course using the vector equation $\vec{r} = \vec{w} + \vec{p}$. Finally, we translate the resultant back into polar form so we can see the speed and heading.

   Then $\vec{w} = \left\langle -30\frac{\sqrt{2}}{2}, 30\frac{\sqrt{2}}{2} \right\rangle = \langle -15\sqrt{2}, 15\sqrt{2} \rangle$

   Similarly, $\vec{p} = \langle 0, -250 \rangle$

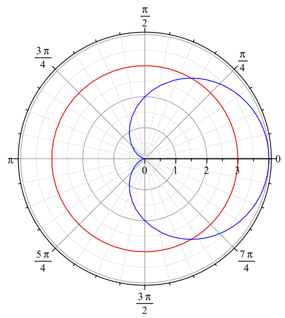
   Then $\vec{r} = \langle -15\sqrt{2}, 15\sqrt{2} - 250 \rangle \approx \langle -21.21, -228.79 \rangle$

   From this, the speed is $\|\vec{r}\| \approx \sqrt{(21.21)^2 + (228.79)^2} \approx 229.77$ miles per hour.

   To find the course, we recall that $tan\theta = \frac{y}{x} \approx \frac{-228.78}{-21.21}$, so $\theta_R \approx 84.7°$.

   Subtracting this from 270° gives a course of 185.3° clockwise from North.

6. Consider the polar functions given by $r = 2 + 2\cos\theta$ and $r = 3$ [See the graph below].



(a) (6 points) Find the points of intersection of these two curves *exactly.*

First we solve the equation $2 + 2\cos\theta = 3$, or $2\cos\theta = 1$. Thus $\cos\theta = \frac{1}{2}$

Hence $\theta = \dfrac{\pi}{3}$ or $\theta = -\dfrac{\pi}{3}\left[= \dfrac{5\pi}{3}\right]$.

Therefore, the points of intersection are: $\left(3, \dfrac{\pi}{3}\right)$ and $\left(3, -\dfrac{\pi}{3}\right)$

(b) (12 points) Set up one or more definite integrals representing the area inside **both** of these polar curves. [You DO NOT need to evaluate your integral, and you may use any obvious symmetry to make setting up yourintegrals more convenient]

The key step in this problem is to notice that we will need to use two integral since these two functions cross at the points of intersection.

In fact, from $\theta = -\frac{\pi}{3}$ to $\theta = \frac{\pi}{3}$ $r = 3$ forms the boundary of the region inside both curves.

However, from $\theta = \frac{\pi}{3}$ to $\theta = \frac{5\pi}{3}$, $r = 2 + 2\cos\theta$ forms the boundary of the region inside both curves.

From this (and using symmetry), we see that $A = 2\displaystyle\int_0^{\frac{\pi}{3}} \frac{1}{2}(3)^2\, d\theta + 2\int_{\frac{\pi}{3}}^{\pi} \frac{1}{2}(2 + 2\cos\theta)^2\, d\theta$