Math 323
Exam 2 - Version 2 **Name:**______________________________

Instructions: You will have 60 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.

1. Given the points $P(-2,1,3)$, $Q(0,2,-1)$ and $R(5,-3,2)$

   (a) (6 points) Find the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$

   First notice that $\overrightarrow{PQ} = \langle 2,1,-4\rangle$ and $\overrightarrow{PR} = \langle 7,-4,-1\rangle$.

   Let θ be the angle between these two vectors.

   Then $\cos\theta = \dfrac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{\|\overrightarrow{PQ}\|\|\overrightarrow{PR}\|} = \dfrac{14-4+4}{\sqrt{4+1+16}\sqrt{49+1+16}} = \dfrac{14}{\sqrt{21}\sqrt{66}}$

   Thus $\theta = \cos^{-1}\left(\frac{14}{\sqrt{21}\sqrt{66}}\right) \approx 67.91°$

   (b) (6 points) Find $comp_{\overrightarrow{PQ}}\overrightarrow{PR}$.

   Recall that $\overrightarrow{PQ} = \langle 2,1,-4\rangle$ and $\overrightarrow{PR} = \langle 7,-4,-1\rangle$.

   Then $comp_{\overrightarrow{PQ}}\overrightarrow{PR} = \dfrac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{\|\overrightarrow{PQ}\|} = \dfrac{14}{\sqrt{21}} = \dfrac{2\sqrt{21}}{3} \approx 3.06$

   (c) (6 points) Find a parametric equation for the line containing P and R.

   We will use the vector $\overrightarrow{PR} = \langle 7,-4,-1\rangle$ and the point $P(-2,1,3)$ [we could have chosen to use any point on the line]. Then:

   $$\ell : \begin{cases} x = -2+7t \\ y = 1-4t \qquad t\in\mathbb{R} \\ z = 3-t \end{cases}$$

   (d) (8 points) Find an equation for the plane containing P, Q, and R.

   Recall that a normal to this plane is given by:

   $$\vec{n} = \overrightarrow{PQ}\times\overrightarrow{PR} = \langle 2,1,-4\rangle\times\langle 7,-4,-1\rangle = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & 1 & -4 \\ 7 & -4 & -1 \end{vmatrix} = \vec{\imath}(-1-16)-\vec{\jmath}(-2+28)+\vec{k}(-8-7) = -17\vec{\imath}-26\vec{\jmath}-15\vec{k}$$

   Then the plane, using $P:(-2,1,3)$, has equation: $-17(x+2)-26(y-1)-15(z-3)=0$, or, simplifying, $17x+26y+15z-37=0$.

2. (10 points) Determine whether the following pair of lines are parallel, skew, or intersecting. If they intersect, find the point of intersection.

$$\ell_1 : \begin{cases} x = 2t \\ y = 3 - t \qquad t \in \mathbb{R} \\ z = 5 - 3t \end{cases} \qquad \ell_2 : \begin{cases} x = 4 \\ y = 2 + s \qquad s \in \mathbb{R} \\ z = 3 + 2s \end{cases}$$

Notice that these lines are not parallel, since their vectors, $\langle 2, -1, -3\rangle$ and $\langle 0, 1, 2\rangle$ are not scalar multiples of each other.

We begin to look for a point of intersection by equating the two x coordinate functions. If $2t = 4$, so $t = 2$.

Using this, equating the y equations and substituting, $3 - t = 2 + s$ becomes $1 - 2 = 2 + s$, so $s = -1$

Evaluating the first line at $t = 2$ gives $(2(2), 3 - 2, 5 - 3(2)) = (4, 1, -1)$. Evaluating the second line at $s = -1$ gives $(4, 2 - 1, 3 + 2(-1)) = (4, 1, 1)$.

Since the z-coordinates do not agree, we see that these lines are skew.

3. (10 points) Find the line of intersection of the planes $x - 3y - z = 2$ and $3x + 2y + z = -1$

These planes are not parallel, since their normal vectors $\langle 1, -3, -1\rangle$ and $\langle 3, 2, 1\rangle$ are not scalar multiples of each other.

Therefore, there must be a line common to both planes.

One way to find the line of intersection is to eliminate the z variable by adding the two planar equations:

$$\begin{array}{l} x - 3y - z = 2 \\ \underline{3x + 2y + z = -1} \\ 4x - y = 1 \end{array}$$

Solving for y in this equation gives $y = 4x - 1$.

Then, substituting this expression for y into our first equation gives:

$x - 3(4x - 1) - z = 2$, or $x - 12x + 3 - z = 2$. Then, solving for z, we have $z = -11x + 1$

Finally, we set $x = t$ to obtain a parameterized equation for the line of intersection: $\ell : \begin{cases} x = t \\ y = -1 + 4t \quad t \in \mathbb{R} \\ z = 1 - 11t \end{cases}$

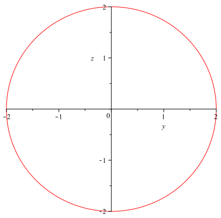
4. Sketch at least 3 non-trivial traces, then sketch and identify the surface given by each of the following equations:

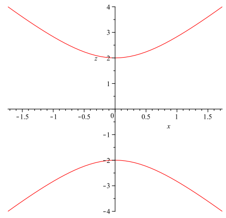
(a) (12 points) $x - y^2 = 4z^2$

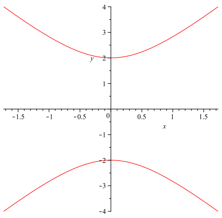
Rewriting this by moving terms gives: $x = y^2 + 4z^2$

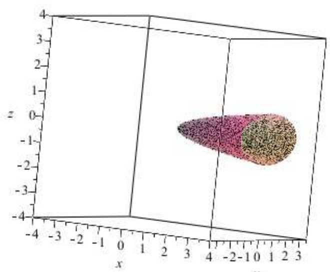
If we set $x = 0$, we get $y^2 + 4z^2 = 0$, which is a single point (and so is a trivial trace, so we must find three other traces).

$y = 0$ gives $x = 4z^2$ $z = 0$ gives $x = y^2$ $x = 4$ gives $y^2 + 4z^2 = 4$ or $\frac{y^2}{4} + z^2 = 1$









The figure is a paraboloid with vertex at the point $(0,0,0)$ opening along
and with elliptical cross sections parallel to the yz-plane.

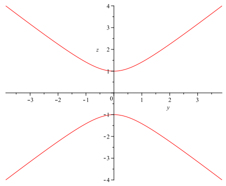
(b) (12 points) $z^2 - 1 = y^2 + 4x^2$

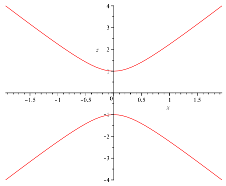
Rewriting this by moving terms gives: $z^2 - y^2 - 4x^2 = 1$.

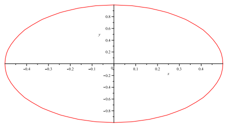
If we set $z = 0$, we get $-y^2 - 4x^2 = 1$ or $y^2 + 4x^2 = -1$, which is empty (and so is a trivial trace, so we must find three other traces).

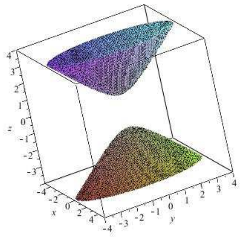
If $|z| = 1$, we get $y^2 + 4x^2 = 0$, which is a single point (and so is a trivial trace, so we must still find three other traces).

$x = 0$ gives $z^2 - y^2 = 1$ $y = 0$ gives $z^2 - 4x^2 = 1$ $|z| = \sqrt{2}$ gives $y^2 + 4x^2 = 1$









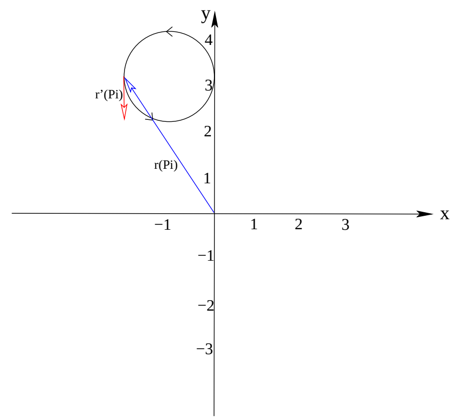
The figure is a hyperboloid of two sheets with vertices at the points $(0,0,1)$ and $(0,0$
and with elliptical cross sections parallel to the xy-plane

5. Let $\vec{r}(t) = \langle \cos t - 1, \sin t + 3\rangle$

(a) (6 points) Sketch the graph of the curve $\mathcal{C}$ traced out by $\vec{r}(t)$ for $0 \le t \le 2\pi$, indicating the orientation of the curve.

We begin by computing a table of values using key inputs for the trigonometric functions. Note that using algebra, if $x = \cos t - 1$ and $y = \sin t + 3$, then $x + 1 = \cos t$ and $y - 3 = \sin t$, so $(x+1)^2 + (y-3)^2 = 1$. Thus the underlying graph is a circle of radius 1 centered at the point $(-1, 3)$.

t	x	y
0	0	3
$\frac{\pi}{2}$	-1	4
π	-2	3
$\frac{3\pi}{2}$	-1	2
2π	0	3



(b) (6 points) Compute the vectors $\vec{r}(\pi)$ and $\vec{r}\,'(\pi)$, then add them to your graph.

From the table above, $\vec{r}(\pi) = \langle -2, 3\rangle$

Notice that $\vec{r}'(t) = \langle -\sin t, \cos t\rangle$, so $\vec{r}\,'(\pi) = \langle 0, -1\rangle$.

See the graph above.

(c) (6 points) Find all values of t on $[0, 2\pi]$ for which $\vec{r}(t)$ and $\vec{r}\,'(t)$ are perpendicular.

Recall that two vectors are parallel if and only if their dot product is zero. Therefore, we will look at the vector equation:
$\vec{r}(t) \cdot \vec{r}\,'(t) = 0$, or $\langle \cos t - 1, \sin t + 3\rangle \cdot \langle -\sin t, \cos t\rangle = 0$

That is, $-\sin t \cos t + \sin t + \sin t \cos t + 3\cos t = \sin t + 3\cos t = 0$. Then $\sin t = -3\cos t$

Therefore, $\frac{\sin t}{\cos t} = -3$, hence $t = \arctan(-3) + k\pi$

Since we are looking for solutions on the interval $[0, 2\pi]$, there are two solutions: $t = \arctan(-3) + \pi$ and $t = \arctan(-3) + 2\pi$.

6. Suppose that an athlete throws a shot put at an angle of 60° to the horizontal at in initial speed of 40 ft/sec, and that it leaves his hand 6 feet above the ground. Recall that gravity is $32 ft/sec^2$. You may assume that gravity is the only force acting on the shot put other than its initial velocity.

(a) (8 points) Find equations for $\vec{a}(t)$, $\vec{v}(t)$, and $\vec{r}(t)$.

$\vec{a}(t) = \langle 0, -32\rangle$.

$\vec{v}(t) = \langle 0, -32t\rangle + \langle 40\cos 60°, 40\sin 60°\rangle = \langle 20, 20\sqrt{3} - 32t\rangle$.

$\vec{r}(t) = \langle 20t, 20\sqrt{3}t - 16t^2\rangle + \langle 0, 6\rangle = \langle 20t, 6 + 20\sqrt{3}t - 16t^2\rangle$.

(b) (6 points) Use the equations you found in part (a) above to compute the horizontal range of the shot put.

The projectile lands when the $\vec{\jmath}$ component of $\vec{r}(t)$ is zero. That is, when $6 + 20\sqrt{3}t - 16t^2 = 0$. Using the quadratic formula, we have $t = \dfrac{-20\sqrt{3} \pm \sqrt{\left(20\sqrt{3}\right)^2 - 4(-16)(6)}}{2(-16)}$. That is, when $t \approx 2.326$ or when $t \approx -0.161$ seconds.
We reject the negative solution and take $t \approx 1.774$.

The horizontal range is found by evaluating the $\vec{\imath}$ component of $\vec{r}(t)$ when $t \approx 2.326$ seconds.
This gives $20(2.326) \approx 46.525$ feet.