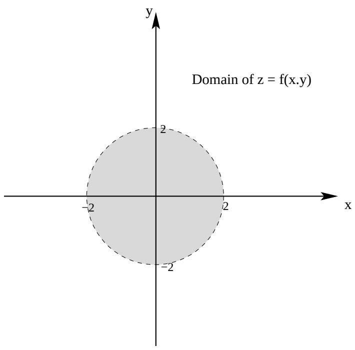
Math 323
Exam 3 - Version 1 **Name:**______________________________

Instructions: You will have 55 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.

1. Let $z = f(x, y) = \ln\left(4 - x^2 - y^2\right)$.

   (a) (8 points) Sketch the domain of this function.

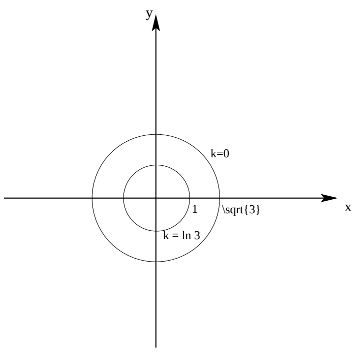
   Notice that to be in the domain of this function, we need $4 - x^2 - y^2 > 0$, or $4 > x^2 + y^2$. Thus the domain of this function is the subset of the xy-plane strictly inside the circle given by the equation $x^2 + y^2 = 4$ (See the graph below).



   (b) (8 points) Sketch contours for $z = f(x, y)$ when $z = 0$ and when $z = \ln 3$ on the same graph.

   When $z = 0$, $f(x, y) = \ln\left(4 - x^2 - y^2\right) = 0$, so $4 - x^2 - y^2 = e^0 = 1$. Thus $x^2 + y^2 = 3$.

   When $z = \ln 3$, $f(x, y) = \ln\left(4 - x^2 - y^2\right) = \ln 3$, so $4 - x^2 - y^2 = e^{\ln 3} = 3$. Thus $x^2 + y^2 = 1$.



2. (10 points) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + 2y^2}$ does not exist.

   Computing the limit along many pairs of paths would show that this limit does not exist. Here is one possible pair:

   Along $x = 0$: $\displaystyle\lim_{(0,y)\to(0,0)} \frac{0}{2y^2} = 0.$

   Along $x = y$: $\displaystyle\lim_{(x,x)\to(0,0)} \frac{2x^2}{x^2 + 2x^2} = \lim_{(x,x)\to(0,0)} \frac{2x^2}{3x^2} = \lim_{(x,x)\to(0,0)} \frac{2}{3} = \frac{2}{3}.$

   Since the value of the limit along these two paths does not agree ($0 \neq \frac{2}{3}$), this limit does not exist.

3. Given the equation: $\sin(xyz) + 5x^2y - 3xy^2 - 2y^3z + 8 = 0$:

(a) (8 points) Use implicit differentiation to find $\dfrac{\partial z}{\partial y}$

Recall that $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}$.

Also, $F_y = xz\cos(xyz) + 5x^2 - 6xy - 6y^2z$ and $F_z = xy\cos(xyz) - 2y^3$.

Then $\dfrac{\partial z}{\partial y} = -\dfrac{xz\cos(xyz) + 5x^2 - 6xy - 6y^2z}{xy\cos(xyz) - 2y^3} = \dfrac{-xz\cos(xyz) - 5x^2 + 6xy + 6y^2z}{xy\cos(xyz) - 2y^3}$

(b) (8 points) Find an equation for the tangent plane to this surface at the point $(1, -1, 0)$.

We first use the gradient to find a normal vector to the tangent plane:

$\nabla F = \langle F_x, F_y, F_z \rangle = \langle yz\cos(xyz) + 10xy - 3y^2, xz\cos(xyz) + 5x^2 - 6xy - 6y^2z, xy\cos(xyz) - 2y^3 \rangle$

Then $\nabla F(1, -1, 0) = \langle 0 + (-10) - 3, 0 + 5 + 6 - 0, -1\cos(0) - (-2) \rangle = \langle -13, 11, 1 \rangle = \vec{n}$

Using $\vec{n} = \langle -13, 11, 1 \rangle$ and the point $P(1, -1, 0)$, the desired tangent plane has the following formula:

$-13(x-1) + 11(y+1) + 1(z-0) = 0$. Simplifying gives $-13x + 13 + 11y + 11 + z = 0$ or $-13x + 11y + z = -24$

4. (10 points) Let $w = f(x, y) = 7x^3y^2$.
Find the *differential* dw and use it to approximate Δw as the input changes from $(1, 2)$ to $(1.2, 1.9)$

First, recall that $dw = f_x(x, y)dx + f_y(x, y)dy$.

Here, $f_x(x, y) = 21x^2y^2$, $f_y(x, y) = 14x^3y$, $dx = \Delta x = 0.2$, $dy = \Delta y = -0.1$, $x = 1$, and $y = 2$.

Then $dw = 21x^2y^2dx + 14x^3ydy = 21(1)(2)^2(0.2) + 14(1)^3(2)(-0.1)$

$= (84)(0.2) + (28)(-0.1) = 16.8 - 2.8 = 14.$

5. (10 points) Suppose $w = f(x, y)$ where $x = 2st^2$ and $y = s - 2t$. Also suppose that $f_x(x, y) = 6xy$ and $f_y(x, y) = 3x^2$. Find the value of $\frac{\partial w}{\partial s}$ when $s = 2$ and $t = -1$.

Recall that $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} = f_x(x, y) \cdot \frac{\partial x}{\partial s} + f_y(x, y) \cdot \frac{\partial y}{\partial s}$

Furthermore, $\frac{\partial x}{\partial s} = 2t^2$, $\frac{\partial y}{\partial s} = 1$, and we are given $f_x(x, y) = 6xy$ and $f_y(x, y) = 3x^2$.

Then $\frac{\partial w}{\partial s} = (6xy)(2t^2) + (3x^2)(1)$. Notice that when $s = 2$ and $t = -1$, then $x = 2(2)(-1)^2 = 4$, and $y = 2 - 2(-1) = 4$.

Substituting, we have $\frac{\partial w}{\partial s} = 6(4)(4) \cdot 2(-1)^2 + 3(4)^2(1) = 192 + 48 = 240$.

6. (10 points) Given that $z = f(x, y, z) = e^{xyz}$ find the derivative of f at the point $(1, 1, 1)$ and in the direction of the vector $\langle -1, 3, 5 \rangle$.

Recall that $D_{\vec{u}} f(a, b, c) = \nabla f(a, b, c) \cdot \vec{u}$. Here, $\nabla f = \langle f_x, f_y, f_z \rangle = \langle yze^{xyz}, xze^{xyz}, xye^{xyz} \rangle$

Then $\nabla f(1, 1, 1) = \langle (1)e^1, (1)e^1, (1)e^1 \rangle = \langle e, e, e \rangle$.

Also, $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{\langle -1, 3, 5 \rangle}{\sqrt{1 + 9 + 25}} = \dfrac{\langle -1, 3, 5 \rangle}{\sqrt{35}} = \langle \dfrac{-1}{\sqrt{35}}, \dfrac{3}{\sqrt{35}}, \dfrac{5}{\sqrt{35}} \rangle$

Then $D_{\vec{u}} f(1, 1, 1) = \nabla f(1, 1, 1) \cdot \vec{u} = \langle e, e, e \rangle \cdot \langle \frac{-1}{\sqrt{35}}, \frac{3}{\sqrt{35}}, \frac{5}{\sqrt{35}} \rangle = \frac{-e}{\sqrt{35}} + \frac{3e}{\sqrt{35}} + \frac{5e}{\sqrt{35}} = \frac{7e}{\sqrt{35}} = \frac{7e\sqrt{35}}{35} = \frac{e\sqrt{35}}{5}$

7. (16 points) Find all the critical points of $f(x, y) = 2x^2 + y^3 - x^2 y - 3y$, and classify them using the Discriminant.

Since $f(x, y)$ is continuous and has continuous first partials (it is a polynomial in two variables), its critical points occur when $f_x = f_y = 0$.

Notice that $f_x(x, y) = 4x - 2xy$ and $f_y(x, y) = 3y^2 - x^2 - 3$.

If $f_x = 0$, then $4x - 2xy = 0$, so $2x(2 - y) = 0$. Hence $x = 0$ or $y = 2$.

If $x = 0$, then if $f_y = 3y^2 - (0)^2 - 3 = 0$, then $3y^2 = 3$, so $y^2 = 1$. Thus $y = \pm 1$, giving critical points $(0, 1)$ and $(0, -1)$.

If $y = 2$, then if $f_y = 3(2)^2 - x^2 - 3 = 0$, then $12 - x^2 - 3 = 0$, so $x^2 = 9$. Thus $x = \pm 3$, giving critical points $(3, 2)$ and $(-3, 2)$.

Next, we classify these critical values using the discriminant. Recall that $D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$.

Notice that $f_{xx} = 4 - 2y$, $f_{yy} = 6y$, and $f_{xy} = -2x$. Then we have the following:

$D(0, 1) = (2)(6) - 0^2 = 12 > 0$, and $f_{xx}(0, 1) = 2 > 0$ so a local minimum occurs at $(0, 1, f(0, 1))$.

$D(0, -1) = (6)(-6) - 0^2 = -36 < 0$, so a saddle point occurs at $(0, -1, f(0, -1))$.

$D(3, 2) = (0)(12) - (-6)^2 = -36 < 0$, so a saddle point occurs at $(3, 2, f(3, 2))$.

$D(-3, 2) = (0)(12) - (6)^2 = -36 < 0$, so a saddle point occurs at $(-3, 2, f(-3, 2))$.

8. (16 points) Use Lagrange multipliers to find the point on the plane $x + 2y - 3z = 1$ closest to the origin.

Notice that we are trying to minimize the distance of a point from the origin. Since it is equivalent (and more convenient) to minimize the **squared** distance from the origin, we will take $f(x, y, z) = x^2 + y^2 + z^2$ to be the function we are trying to minimize.

Next, we see that the point in question must be a point on the plane $x + 2y - 3z = 1$. Therefore, we will take $g(x, y, z) = x + 2y - 3z - 1 = 0$ to be the constraint equation.

Using Lagrange's method, we recall that solutions occur when $\nabla f = \lambda \nabla g$ and at points where the constraint equation $g(x, y) = 0$ is satisfied.

We see that $\nabla f = \langle f_x, f_y, f_z \rangle = \langle 2x, 2y, 2z, \rangle$ and $\nabla g = \langle g_x, g_y, g_z \rangle = \langle 1, 2, -3 \rangle$

Then we have $2x = \lambda \ 2y = 2\lambda$ and $2z = -3\lambda$. Then $\lambda = 2x = y = -\frac{2}{3}z$.

But then $y = 2x$, and $2x = -\frac{2}{3}z$, or $z = -\frac{3}{2}(2x) = -3x$.

Substituting into the constraint equation gives $x + 2(2x) - 3(-3x) = 1$, or $x + 4x + 9x = 1$.

Then $14x = 1$, so $x = \frac{1}{14}$. Then $y = \frac{2}{14} = \frac{1}{7}$, and $z = -\frac{3}{14}$.

We know this this solution must be a minimum, since points on any given plane can be chosen so that they are arbitrarily far from the origin. Hence the closest point to the origin on the plane $x + 2y - 3z = 1$ is the point $\left(\dfrac{1}{14}, \dfrac{1}{7}, -\dfrac{3}{14} \right)$