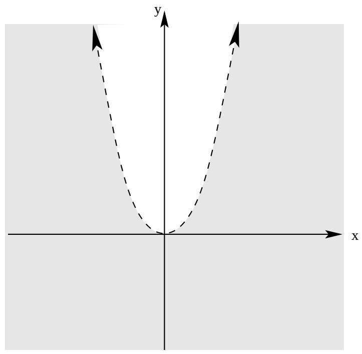
Math 323
Exam 3 - Version 2 **Name:**______________________________

Instructions: You will have 55 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.

1. Let $z = f(x, y) = \dfrac{1}{\sqrt{x^2 - y}}$.

   (a) (8 points) Sketch the domain of this function.

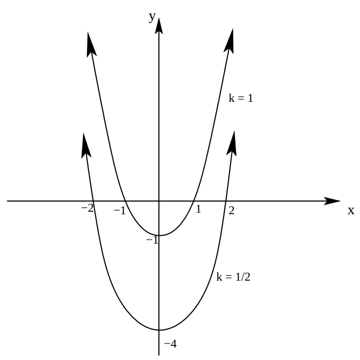
   Notice that to be in the domain of this function, we need $x^2 - y > 0$, or $x^2 > y$. Thus the domain of this function is the subset of the xy-plane strictly below (that points with smaller y-coordinates than the points on) the parabola given by the equation $y = x^2$ (See the graph below).



   (b) (8 points) Sketch contours for $z = f(x, y)$ when $z = 1$ and when $z = \frac{1}{2}$.

   When $z = 1$, $f(x, y) = \dfrac{1}{\sqrt{x^2 - y}} = 1$, so $\sqrt{x^2 - y} = 1$. Thus $x^2 - y = 1^2 = 1$, or $y = x^2 - 1$.

   When $z = \frac{1}{2}$, $f(x, y) = \dfrac{1}{\sqrt{x^2 - y}} = \dfrac{1}{2}$, so $\sqrt{x^2 - y} = 2$. Thus $x^2 - y = 2^2 = 4$, or $y = x^2 - 4$.



2. (10 points) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4 + y^2}$ does not exist.

   We will compute the limit along two paths and obtain different values in order to show that this limit does not exist. In this example, we need to choose our second path carefully using the form of the expression to guide our choice.

   Along $x = 0$: $\displaystyle\lim_{(0,y)\to(0,0)} \frac{0}{y^2} = 0$.

   Along $y = x^2$: $\displaystyle\lim_{(x,x^2)\to(0,0)} \frac{2x^2(x^2)}{x^4 + (x^2)^2} = \lim_{(x,x^2)\to(0,0)} \frac{2x^4}{2x^4} = \lim_{(x,x^2)\to(0,0)} \frac{2}{2} = 1$.

   Since the value of the limit along these two paths does not agree ($0 \neq 1$), this limit does not exist.

3. Given the equation: $\sin(xyz) + 5x^2y - 3xy^2 - 2y^3z - 8 = 0$:

   (a) (8 points) Use implicit differentiation to find $\dfrac{\partial z}{\partial x}$

   Recall that $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$.

   Also, $F_x = yz\cos(xyz) + 10xy - 3y^2$ and $F_z = xy\cos(xyz) - 2y^3$.

   Then $\dfrac{\partial z}{\partial x} = -\dfrac{yz\cos(xyz) + 10xy - 3y^2}{xy\cos(xyz) - 2y^3} = \dfrac{-z\cos(xyz) - 10x + 3y}{x\cos(xyz) - 2y^2}$

   (b) (8 points) Find an equation for the tangent plane to this surface at the point $(-1, 1, 0)$.

   We first use the gradient to find a normal vector to the tangent plane:

   $\nabla F = \langle F_x, F_y, F_z \rangle = \langle yz\cos(xyz) + 10xy - 3y^2, xz\cos(xyz) + 5x^2 - 6xy - 6y^2z, xy\cos(xyz) - 2y^3 \rangle$

   Then $\nabla F(-1, 1, 0) = \langle 0 + (-10) - 3, 0 + 5 + 6 - 0, -1\cos(0) - (2) \rangle = \langle -13, 11, -3 \rangle = \vec{n}$

   Using $\vec{n} = \langle -13, 11, -3 \rangle$ and the point $P(-1, 1, 0)$, the desired tangent plane has the following formula:

   $-13(x+1) + 11(y-1) - 3(z-0) = 0$. Simplifying gives $-13x - 13 + 11y - 11 - 3z = 0$ or $-13x + 11y - 3z = 24$

4. (10 points) Let $w = f(x, y) = 10x^2y^3$.
   Find the *differential* dw and use it to approximate Δw as the input changes from $(1, 2)$ to $(0.9, 2.2)$

   First, recall that $dw = f_x(x, y)dx + f_y(x, y)dy$.

   Here, $f_x(x, y) = 20xy^3$, $f_y(x, y) = 30x^2y^2$, $dx = \Delta x = -0.1$, $dy = \Delta y = 0.2$, $x = 1$, and $y = 2$.

   Then $dw = 20xy^3dx + 30x^2y^2dy = 20(1)(2)^3(-0.1) + 30(1)^2(2)^2(0.2)$

   $= (160)(-0.1) + (120)(0.2) = -16 + 24 = 8$.

5. (10 points) Suppose $w = f(x, y)$ where $x = 3s^2t$ and $y = t - 2s$. Also suppose that $f_x(x, y) = 5y^2$ and $f_y(x, y) = 10xy$. Find the value of $\dfrac{\partial w}{\partial s}$ when $s = 2$ and $t = -1$.

Recall that $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} = f_x(x, y) \cdot \frac{\partial x}{\partial s} + f_y(x, y) \cdot \frac{\partial y}{\partial s}$

Furthermore, $\frac{\partial x}{\partial s} = 6st$, $\frac{\partial y}{\partial s} = -2$, and we are given $f_x(x, y) = 5y^2$ and $f_y(x, y) = 10xy$.

Then $\frac{\partial w}{\partial s} = (5y^2)(6st) + (10xy)(-2)$. Notice that when $s = 2$ and $t = -1$, then $x = 3(2)^2(-1) = -12$, and $y = -1 - 2(2) = -5$.

Substituting, we have $\frac{\partial w}{\partial s} = 5(-5)^2 \cdot 6(2)(-1) + 10(-12)(-5)(-2) = -1500 - 1200 = -2700$.

6. (10 points) Given that $z = f(x, y, z) = ye^{xz}$ find the derivative of f at the point $(1, -1, 0)$ and in the direction of the vector $\langle 5, -1, 3 \rangle$.

Recall that $D_{\vec{u}}f(a, b, c) = \nabla f(a, b, c) \cdot \vec{u}$. Here, $\nabla f = \langle f_x, f_y, f_z \rangle = \langle yze^{xz}, e^{xz}, xye^{xz} \rangle$

Then $\nabla f(1, -1, 0) = \langle (-1)(0)e^0, e^0, (1)(-1)e^0 \rangle = \langle 0, 1, -1 \rangle$.

Also, $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{\langle 5, -1, 3 \rangle}{\sqrt{25 + 1 + 9}} = \dfrac{\langle 5, -1, 3 \rangle}{\sqrt{35}} = \langle \dfrac{5}{\sqrt{35}}, -\dfrac{1}{\sqrt{35}}, \dfrac{3}{\sqrt{35}} \rangle$

Then $D_{\vec{u}}f(1, -1, 0) = \nabla f(1, -1, 0) \cdot \vec{u} = \langle 0, 1, -1 \rangle \cdot \langle \frac{5}{\sqrt{35}}, -\frac{1}{\sqrt{35}}, \frac{3}{\sqrt{35}} \rangle = (0)\frac{5}{\sqrt{35}} - (1)\frac{1}{\sqrt{35}} - \frac{3}{\sqrt{35}} = \frac{-4}{\sqrt{35}} = -\frac{4\sqrt{35}}{35}$

7. (16 points) Find all the critical points of $f(x, y) = 4xy - x^4 - y^4$, and classify them using the Discriminant.

Since $f(x, y)$ is continuous and has continuous first partials (it is a polynomial in two variables), its critical points occur when $f_x = f_y = 0$.
Notice that $f_x(x, y) = 4y - 4x^3$ and $f_y(x, y) = 4x - 4y^3$.

If $f_x = 0$, then $4y = 4x^3$, so $y = x^3$. Similarly, if $f_x = 0$, then $4x = 4y^3$, so $x = y^3$

Substituting the second equation into the first equation gives: $y = (y^3)^3 = y^9$, or $y^9 - y = 0$.

Then $y(y^8 - 1) = 0$, so $y = 0$, or $y^8 = 1$, in which case, $y = \pm 1$.

Combining this with the fact that $x = y^3$ gives the critical points: $(0, 0)$, $(1, 1)$, and $(-1, -1)$

Next, we classify these critical values using the discriminant. Recall that $D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$.

Notice that $f_{xx} = -12x^2$, $f_{yy} = -12y^2$, and $f_{xy} = 4$. Then we have the following:

$D(0, 0) = (0)(0) - 4^2 = -16 < 0$, so a saddle point occurs at $(0, 0, f(0, 0))$.

$D(1, 1) = (-12)(-12) - 4^2 = 144 - 16 = 128 > 0$, and $f_{xx}(0, 1) == 12 < 0$ so a local maximum occurs at $(1, 1, f(1, 1))$.

$D(-1, -1) = (-12)(-12) - 4^2 = 144 - 16 = 128 > 0$, and $f_{xx}(0, 1) == 12 < 0$ so a local maximum occurs at $(-1, -1, f(-1, -1))$.

8. (16 points) Use Lagrange multipliers to solve for following: The sum of three positive numbers x, y, and z is 120. Find the maximum possible value for xy^2z.

   Notice that we are trying to maximize the value of the function $f(x, y, z) = xy^2z$.

   Since the sum of the number must be 120, we will take $g(x, y, z) = x + y + z - 120 = 0$ to be the constraint equation.

   Using Lagrange's method, we recall that solutions occur when $\nabla f = \lambda \nabla g$ and at points where the constraint equation $g(x, y) = 0$ is satisfied.

   We see that $\nabla f = \langle f_x, f_y, f_z \rangle = \langle y^2z, 2xyz, xy^2, \rangle$ and $\nabla g = \langle g_x, g_y, g_z \rangle = \langle 1, 1, 1 \rangle$

   Then we have $y^2z = \lambda$, $2xyz = \lambda$ and $xy^2 = \lambda$. Then $\lambda = y^2z = 2xyz = xy^2$.

   Consider $y^2z = 2xyz$. Then either $y = 0$, $z = 0$, or $y = 2x$.

   Similarly, consider $2xyz = xy^2$. Then either $x = 0$, or $2z = y$.

   Notice that if any of the variables x, y, or z are zero, then $f(x, y, z) = xy^2z = 0$. Then we will consider the non-zero equivalences.

   If $y = 2x = 2z$, then $y = 2x$, and $z = x$. We substitute into the constraint equation, yielding $x + 2x + x = 120$, or $4x = 120$.

   Then $x = 30$, $z = x = 30$, and $y = 2x = 60$.

   Notice that $f(30, 60, 30) = (30)(60)^2(30) = 3,240,000$, which is greater then zero. Hence the maximum value for the expression xy^z under the constraint $x + y + z = 120$ is $3,240,000$.