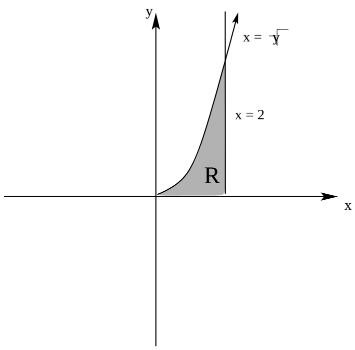
Math 323
Exam 4 - Version 2 **Name:**________________________________

Instructions: You will have 55 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.

1. Given the integral $\int_0^4 \int_{\sqrt{y}}^2 \cos\left(x^3\right)\, dx\, dy$

   (a) (6 points) Graph the region of integration R.

   Using the bounds on the integral given, we have $0 \leq y \leq 4$, and $\sqrt{y} \leq x \leq 2$. Then the graph of the region of integration R is as follows:



   (b) (12 points) Evaluate the integral by changing the order of integration.

   From the graph above, if we switch the order of integration, we have bounds $0 \leq y \leq x^2$ and $0 \leq x \leq 2$. Then the integral is:

$$\int_0^2 \int_0^{x^2} \cos(x^3)\, dy\, dx = \int_0^2 y \cdot \cos(x^3) \Big|_0^{x^2} dx$$

$$= \int_0^2 x^2 \cos(x^3) - 0\, dx = \frac{1}{3}\sin(x^3)\Big|_0^2 = \frac{1}{3}\sin(8) - \frac{1}{3}\sin(0) = \frac{\sin 8}{3}$$

2. (12 points) Let a volume integral be defined by $V = \int_0^2 \int_0^{4-2y} \int_0^{1-\frac{x}{4}-\frac{y}{2}} dz\, dx\, dy$

   Express this integral in rectangular coordinates in the order $dy\, dx\, dz$.

   Notice that using the boundaries above, we have $0 \leq x \leq 4 - 2y - 4z$, $0 \leq y \leq 2 - 2z$, and $0 \leq z \leq 1$.

   Considering the first variable, the "top" surface is the planar equation $z = 1 - \frac{x}{4} - \frac{y}{2}$, or $4z = 4 - x - 2y$. In the xy-plane, this has trace $x + 2y = 4$ or $x = 4 - 2y$, so we see that the solid of integration is the tetrahedron in the first octant formed by the plane above and the coordinate planes.

   From this, we first solve the original plane equation for y, giving $2y = 4 - x - 4z$, or $y = 2 - \frac{x}{2} - 2z$. The trace in the xz-plane is $x + 4z = 4$, or, solving for x, $x = 4 - 4z$. Finally, the z intercept is 1, so we have the following integral:

$$\int_0^1 \int_0^{4-4z} \int_0^{2-\frac{x}{2}-2z} 1\, dy\, dx\, dz$$

3. A lamina occupies the plane region R bounded by $x^2 + y^2 = 1$, and has density function $\delta(x, y) = e^{-\sqrt{x^2+y^2}}$ in kg/cm^2.
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(a) (15 points) Find the mass if this lamina.

We begin by noting that this integral is best evaluated in polar form, so we translate $\delta(x, y) = e^{-\sqrt{x^2+y^2}}$ into $\delta(r, \theta) = e^{-r}$. Next, we see that the polar region is the disk or radius 1 in the plane.

From this, we have the following integral: $\displaystyle\int_0^{2\pi} \int_0^1 e^{-r} \cdot r\, dr\, d\theta.$

From the form of this integral, we see that we will need to use integration by parts in order to evaluate the inner integral. To do this, we take $u = r$ and $dv = e^{-r}\, dr$. Then $du = 1\, dr$ and $v = -e^{-r}$. Then we have the integral:

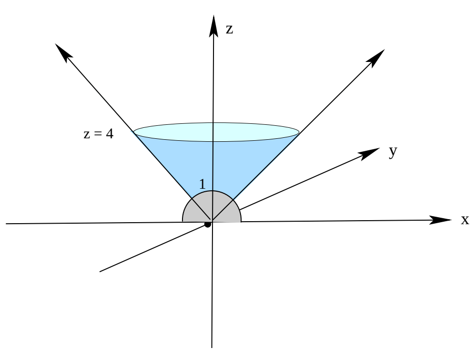
$$\int_0^{2\pi} \int_0^1 e^{-r} \cdot r \sin\theta\, dr\, d\theta = \int_0^{2\pi} -re^{-r}\Big|_0^1 + \int_0^{2\pi} \int_0^1 e^{-r}\, dr\, d\theta = \int_0^{2\pi} -re^{-r} - e^{-r}\Big|_0^1 d\theta$$
$$= \int_0^{2\pi} \frac{-1}{e} + \frac{-1}{e} - (0 - 1)\, d\theta = \int_0^{2\pi} 1 - \frac{2}{e}\, d\theta = \left(1 - \frac{2}{e}\right)\Big|_0^{2\pi} = 2\pi\left(1 - \frac{2}{e}\right) \approx 1.66 \text{ kg.}$$

(b) (5 points) Set up an integral representing the moment of this lamina with respect to the y-axis. You DO NOT need to evaluate this integral.

Recall that $M_y = \displaystyle\iint_R x \cdot \delta(x, y)\, dA$. Then, reverting to polar coordinates to get a simpler integral, we have $M_y = \displaystyle\int_0^{2\pi} \int_0^1 r\cos\theta \cdot e^{-r} \cdot r\, dr\, d\theta = \int_0^{2\pi} \int_0^1 r^2 e^{-r} \cos\theta\, dr\, d\theta$

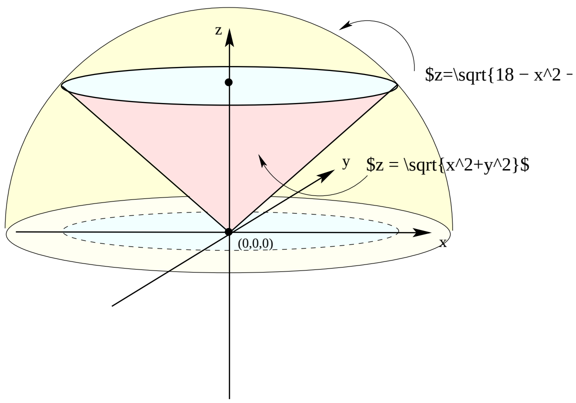
4. (12 points) Given the triple integral $\displaystyle\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_1^{4\sec\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$, sketch the solid Q which serves and the region of integration.

Since we have bounds $0 \le \theta \le 2\pi$, $0 \le \phi \le \frac{\pi}{4}$, and $1 \le \rho \le 4\sec\phi$, the region integration is bounded inside by the sphere of radius 1 centered at the origin. It is bounded outside by $\rho = 4\sec\phi$, which is equivalent to the plane $z = 4$. The region is also within a 45° cone that would normally sit with its cone point at the origin. The graph of this region is as follows:



5. Let Q be the region between $z = \sqrt{18 - x^2 - y^2}$ and $z = \sqrt{x^2 + y^2}$, and suppose $f(x, y, z) = yz$.

First notice that $z = \sqrt{18 - x^2 - y^2}$ is a hemisphere of radius $\sqrt{18} = 3\sqrt{2}$ and that $z = \sqrt{x^2 + y^2}$ is a 45° cone. The intersection of these surfaces is given by $\sqrt{18 - x^2 - y^2} = \sqrt{x^2 + y^2}$ or $18 - x^2 + y^2 = x^2 + y^2$. Then $2x^2 + 2y^2 = 18$, or $x^2 = y^2 = 9$, a circle of radius 3, which serves as the boundary for the region in the plane when integrating in either cylindrical or rectangular coordinates. The graph of this solid is as follows:



(a) (14 points) Set up a triple integral which gives $I = \iiint_Q f(x, y, z)\, dV$ in rectangular coordinates.
DO NOT EVALUATE THE INTEGRAL.

Applying the discussion above and the graph of the region, we see that the integration order $dz\, dy\, dx$ is probably the simplest to use (actually, the order of x and y do not matter)

We then have $\sqrt{x^2 + y^2} \le z \le \sqrt{18 - x^2 - y^2}$, $-\sqrt{9 - x^2} \le y \le \sqrt{9 - x^2}$, and $-3 \le x \le 3$. Therefore, an integral in rectangular coordinates representing I is:

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} yz\, dz\, dy\, dx$$

(b) (14 points) Set up a triple integral which gives $I = \iiint_Q f(x, y, z)\, dV$ in cylindrical coordinates.
DO NOT EVALUATE THE INTEGRAL.

Translating the surfaces and region into cylindrical form, we have: $\sqrt{x^2 + y^2} = \sqrt{r^2} = r$ and $\sqrt{18 - x^2 - y^2} = \sqrt{18 - r^2}$, so the limits of integration will be given by $r \le z \le \sqrt{18 - r^2}$, $0 \le r \le 3$, and $0 \le \theta \le 2\pi$. We also must translate the integrand into cylindrical coordinates. Since $y = r \sin\theta$, we have the following cylindrical integral representing I:

$$\int_0^{2\pi} \int_0^3 \int_r^{\sqrt{18-r^2}} r \sin\theta \cdot z \cdot r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^3 \int_r^{\sqrt{18-r^2}} r^2 (\sin\theta) z\, dz\, dr\, d\theta$$

(c) (14 points) Set up a triple integral which gives $I = \iiint_Q f(x, y, z)\, dV$ in spherical coordinates.
DO NOT EVALUATE THE INTEGRAL.

To translate into spherical coordinates, we notice that the top boundary can be rewritten as $\rho = \sqrt{18} = 3\sqrt{2}$. Also notice that the inner boundary is $\rho = 0$ since the cone sits on top of the origin. Therefore, we have the following bounds: $0 \le \rho \le 3\sqrt{2}$, $0 \le \phi \le \frac{\pi}{4}$, and $0 \le \theta \le 2\pi$.
We also must convert the integrand using $y = \rho \sin\phi \sin\theta$ and $z = \rho\cos\phi$ and the spherical differential $dV = \rho^2 \sin\phi$. Hence a spherical integral representing I is given by:

$$\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{3\sqrt{2}} \rho^4 \sin^2\phi \cos\phi \sin\theta\, d\rho\, d\phi\, d\theta$$