Math 323
Increments and Differentials

Recall: Let $f(x, y)$ be a function of two variables. Last time, we looked at the following formal definitions for partial derivatives with respect to x and y:

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

(provided these limits exist).

Definition: $w = f(x, y)$ and let Δx and Δy be increments of x and y respectively. The *increment* Δw of $w = f(x, y)$ is:

$$\Delta w = f(x + \Delta x, y + \Delta y) - f(x, y)$$

Conceptually, Δw represents the change in the value of the function $f(x, y)$ as the input changes from the point $P(x, y)$ to the point $Q(x + \Delta x, y + \Delta y)$. We generally think of the increments Δx and Δy as being "small".

Example: Let $w = f(x, y) = 3xy - y^2$.

1. Find an algebraic expression for Δw.

   $\Delta w =$

2. Use the increment Δw to find the change in $f(x, y)$ as the input changes from $(1, 1)$ to $(1.1, 0.8)$.

   $\Delta w =$

Definition: The **differentials** of the function $w = f(x, y)$ are as follows:

- $dx = \Delta x$
- $dy = \Delta y$
- $dw = f_x(x, y)\, dx + f_y(x, y)\, dy$

Conceptually, the differential dw is used to *approximate* the value of the increment Δw.

Example: Let $w = f(x, y) = 3xy - y^2$. Find the differential dw.

Theorem 16.14 Let $w = f(x, y)$. Suppose f is defined on a rectangular region $R = \{(x, y) : a < x < b, c < y < d\}$. Also suppose that f_x and f_y are continuous and a point (x_0, y_0) in R. If $(x_0 + \Delta x, y_0 + \Delta y)$ is in R, and $\Delta w = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$, then

$$\Delta w = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and ε_2 are functions of Δx and Δy whose limits approach zero as $(\Delta x, \Delta y) \to (0, 0)$.

Example: Let $w = f(x, y) = 2x^2 - 3xy + 2$. Find the decomposition of $f(x, y)$ guaranteed for by Theorem 16.14.

Definition: A function $w = f(x, y)$ is **differentiable** at the point (x_0, y_0) if Δw can be expressed in the form:

$$\Delta w = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and ε_2 are functions of Δx and Δy whose limits approach zero as $(\Delta x, \Delta y) \to (0, 0)$.

Note: Using this definition and the computations we performed above, we can conclude that the function $w = f(x, y) = 2x^2 - 3xy + 2$ is differentiable at every point in the xy-plane.

Theorem 16.17 If $w = f(x, y)$ and if f_x and f_y are continuous on a rectangular region R, then f is differentiable on R.

Theorem 16.18 If $w = f(x, y)$ is differentiable at (x_0, y_0), then $f(x, y)$ is also continuous at (x_0, y_0).

Note: We can extend the idea of increments and differentials to functions of three variables. If we let $w = f(x, y, z)$ and Δx, Δy, and Δz be increments of x, y, and z respectively, then $\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$ can be written as:

$$\Delta w = f_x(x, y, z)\Delta x + f_y(x, y, z)\Delta y + f_z(x, y, z)\Delta z + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y + \varepsilon_3 \Delta z$$

where where ε_1, ε_2 and ε_3 are functions of Δx, Δy and Δz whose limits approach zero as $(\Delta x, \Delta y, \Delta z) \to (0, 0, 0)$.

Example: Let $f(x, y, z) = x^2y - xyz + z^2$. Use differentials to approximate the change in w as the input changes from $(1, 1, 1)$ to $(1.1, 1.2, 0.9)$