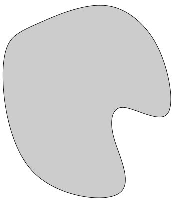
Math 323
Independence of Path

Definitions:
• A region $D \subset \mathbb{R}^n$ (for $n \geq 2$) is called **connected** if every pair of points in D can be connected by a piecewise smooth curve lying entirely in D.

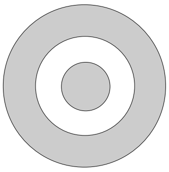
Examples:

D1



Connected

D2



Not Connected

• Let $\mathcal{C}$ be a piecewise smooth path from P to Q contained in a open connected region D. A line integral $\int_{\mathcal{C}} \vec{F} \cdot dr$ is **independent of path** if the integral has the same value along *any* piecewise smooth path from P to Q in D.

Theorem: Let $\vec{F}(x, y) = \langle M(x, y), N(x, y)\rangle$ be a continuous vector field on an open, connected region $D \subset \mathbb{R}^2$. Then the line integral $\int_{\mathcal{C}} \vec{F} \cdot dr$ is independent of path in D if and only if the vector field $\vec{F}$ is conservative. That is, $\vec{(}F)(x, y) = \nabla f(x, y)$ for some scalar function f.

Proof: See pages 982-984 in your text.

Theorem: The Fundamental Theorem of Line Integrals Let $\vec{F}(x, y) = \langle M(x, y), N(x, y)\rangle$ be a continuous vector field on an open, connected region $D \subset \mathbb{R}^2$. Let $\mathcal{C}$ be any piecewise smooth curve in D with initial point (x_1, y_1) and terminal point (x_2, y_2). If $\vec{F}$ is conservative, with $\vec{(}F)(x, y) = \nabla f(x, y)$, then the line integral $\int_{\mathcal{C}} \vec{F} \cdot dr = f(x_2, y_2) - f(x_1, y_1)$.

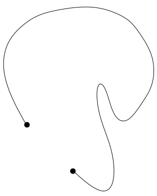
Example: Let $\vec{F}(x, y) = \langle 2xy, x^2 - 4y\rangle$, and let $\mathcal{C}$ be a piecewise smooth path from $P(0, 2)$ to $Q(4, 10)$. Evaluate $\int_{\mathcal{C}} \vec{F} \cdot dr$.

First notice that if $f(x, y) = x^2y - 2y^2$, then $f_x = 2xy$, and $f_y = x^2 - 4y$, so $\vec{F}(x, y) = \nabla f(x, y)$, and thus $\vec{F}$ is a conservative vector field. Moreover, the coordinate functions $f_x = 2xy$ and $f_y = x^2 - 4y$ are continuous. Therefore, using the fundamental theorem of line integrals, we have:

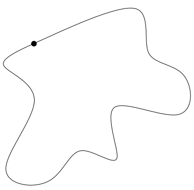
$$\int_{\mathcal{C}} \vec{F} \cdot dr = f(4, 10) - f(0, 2) = [4^2(10) - 2(10)^2] - [0^2(2) - 2(2)^2] = [160 - 200] - [-8] = -32.$$

Definitions:
• A curve $\mathcal{C}$ is **closed** if its beginning and ending points are the same.
• A region D is **simply connected** if every closed curve in D only encloses points also in D.

Examples:

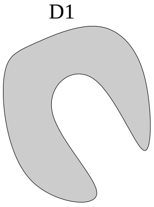


Not Closed



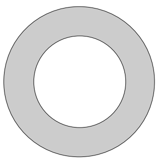
Closed

D1



Simply Connected

D2



Connected
But Not Simply Connected

Theorem: Let $\vec{F}(x, y)$ be a continuous vector field on an open, connected region $D \subset \mathbb{R}^2$. Then $\vec{F}$ is conservative if and only if $\int_{\mathcal{C}} \vec{F} \cdot dr = 0$ for every piecewise smooth closed curve in D.

Proof:

Theorem: Let $\vec{F}(x, y) = \langle M(x, y), N(x, y)\rangle$. If $M(x, y)$ and $N(x, y)$ have continuous first order partial derivatives on a simply connected region $D \subset \mathbb{R}^2$, the the line integral $\int_{\mathcal{C}} M(x, y)\, dx + N(x, y)\, dy$ is independent of path if and only if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Examples:

1. Show that the integral $\int_{\mathcal{C}} 2x\sin(y)\, dx + x^2 cos(y)\, dy$ is independent of path.

   Here, $M(x, y) = 2x\sin(y)$ and $N(x, y) = x^2\cos(y)$. Then $\frac{\partial M}{\partial y} = 2x\cos(y)$ while $\frac{\partial N}{\partial x} = 2x\cos(y)$. But then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, hence this integral is independent of path.

2. Show that the integral $\int_{\mathcal{C}} 2xy\, dx - x^2\, dy$ is **not** independent of path.

   Here, $M(x, y) = 2xy$ and $N(x, y) = -x^2$. Then $\frac{\partial M}{\partial y} = 2x$ while $\frac{\partial N}{\partial x} = -2x$. But then $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, hence this integral is **not** independent of path.

Theorem: The Fundamental Theorem of Line Integrals [3D Version] Let $\vec{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z)\rangle$ be a continuous vector field on an open, connected region $D \subset \mathbb{R}^3$. Let $\mathcal{C}$ be any piecewise smooth curve in D with initial point (x_1, y_1, z_1) and terminal point (x_2, y_2, z_2). If $\vec{F}$ is conservative, with $\vec{(F)}(x, y, z) = \nabla f(x, y)$, then the line integral $\int_{\mathcal{C}} \vec{F} \cdot dr = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$.

Note: A vector field, $\vec{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z)\rangle$ is independent of path if and only if $F(x, y, z) = \nabla f(x, y, z)$ for some scalar function f. Moreover, if F is conservative, and each coordinate function has continuous first partial derivatives, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$, and $\frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}$.

Definition: Let $\vec{F}(x, y, z)$ be a conservative vector field with potential function f. Then the **potential energy** $p(x, y, x)$ of a particle at the point (x, y, x) is given by $p(x, y, x) = -f(x, y, z)$.

Proof: Notice that $\vec{F}(x, y, x) = \nabla f(x, y, x) = -\nabla p(x, y, z)$. Then the work required to move a particle from A to B through this vector field is given by $W = \int_A^B \vec{F} \cdot dr = -p(B) - (-p(A)) = p(A) - p(B)$. In particular, of $p(B) = 0$, then $W = p(A)$.

The Law of Conservation of Energy: If a particle moves from one point to another in a conservative vector field, then the sum of the potential and kinetic energies remains constant throughout the movement of the particle. That is, $p(A) + k(A) = p(B) + k(B)$.