Math 323
Triple Integrals

Goal: Our new goal is to extend the concept of the definite integral to functions of *three* variables. We can interpret these integral as the *hyper volume* "under" this function of three variables and "over" a Volume Q, or, if we take $w = f(x, y, z)$ as a density function, we can also interpret this as finding the mass of the solid Q. We will begin by partitioning Q into an inner partition of rectangular solids. We then approximate the "hyper volume" "under" the function and "over" the solid Q by computing the sum of the volume of the each rectangular solid in the partition multiplied by the value of the function $f(x, y, z)$ at some point within a given rectangular solid.

Definition: Let f be a function of three variables defined on a solid Q. The **triple integral of f over Q**, denoted by $\displaystyle\iiint\limits_Q f(x, y, z)\ dV = \lim_{\|\mathcal{P}\|\to 0} \sum_{Q_{i,j,k}} f(u_i, v_j, w_k)\Delta V_{i,j,k}$, provided the limit exists. When this limit exists, we say that the function $f(x, y)$ is **integrable** over the solid Q.

Definition 17.21: Let Q be a solid, and let $\delta(x, y, z)$ be a function that gives the density at each (x, y, z) in a solid Q. Then the **mass** m of the solid Q is: $m = \displaystyle\iiint\limits_Q \delta(x, y, z)\ dV$

To evaluate triple integrals, we will once again make use of *iterated integrals*. Notice that since there are three variables involved, there are three different ways to "partially integrate" a given function and so there are six potential ways that we can order our iterated integrals. As before, for an integrable function, all possible orders end up having the same value (although some may be easier than others to actually evaluate).

Theorem 17.19 and 17.20 (Fubini):
(I) Let Q be a solid bounded "horizontally" by a pair of surfaces $k_1(x, y)$ and $k_2(x, y)$ each sitting "over" a region R in the plane and suppose that $f(x, y, z)$ is continuous on Q, then

$$\iiint\limits_Q f(x, y, z)\ dV = \iint_R \int_{k_1(x,y)}^{k_2(x,y)} f(x, y, z)\ dz\, dA$$

(II) Further suppose that the region R is bounded in the plane by functions $h_1(x)$ and $h_2(x)$ for x in the interval $[a, b]$. Then:

$$\iiint\limits_Q f(x, y, z)\ dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{k_1(x,y)}^{k_2(x,y)} f(x, y, z)\ dz\ dy\, dx$$

Note: Similar results hold for all six possible ways of ordering the variables when partially antidifferentiating.

Examples:

1. Evaluate the triple integral $\displaystyle\int_0^1 \int_0^1 \int_0^2 4x - 2y + z\, dy\, dx\, dz$.

2. Let Q be the tetrahedron bounded by $x + 2y + z = 2$ and the coordinate planes and suppose that the density of this solid is given by $\delta(x, y, z) = 4yz$. Find the mass of the solid Q.

3. Find the volume of the solid bounded by $1 - x^2$, $z = x^2 - 1$, $y = 2$, and $y = 4$.