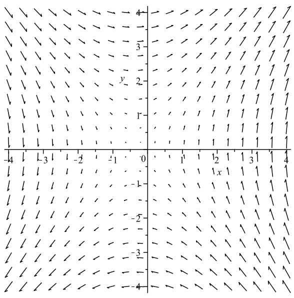
Math 323
Vector Fields

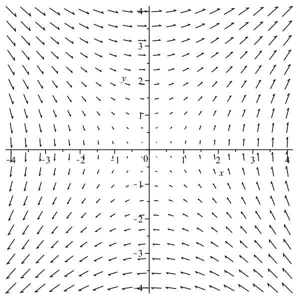
Definitions:
• A *vector field* in the plane is a function $F(x,y) = \langle f_1(x,y), f_2(x,y) \rangle$ that maps the plane into the set of 2-component vectors.
• A *vector field* in 3-space is a function $F(x,y,z) = \langle f_1(x,y,z), f_2(x,y,z), f_3(x,y,z) \rangle$ that maps the plane into the set of 2-component vectors.
• 2D vector fields can be given in either rectangular or polar coordinates. 3D vector fields can be given in rectangular, cylindrical, or shperical coordinates.
• When we evaluate a vector field, the input a point in the plane (or a point in 3-space) and the output is a vector with 2 components (or 3 components in the 3D case). We think of this vector as representing a force acting at the point (wind, electrostatic, etc).
• To form the **graph** of a vector field, several points are input into the defining function, and the resulting vectors are plotted as position vectors starting at the input value, and pointing in the direction given by the correcponding output vector. We generally do not attempt to reflect the magnitude of the output vector, only the direction.

Examples:

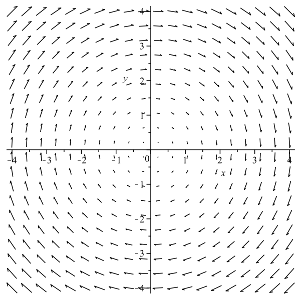
$F(x,y) = \langle 2y, 3x+1 \rangle$



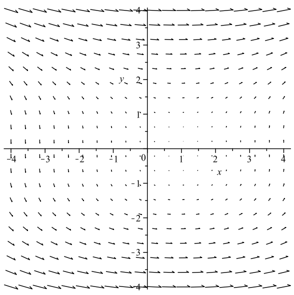
$F(x,y) = \langle y, x \rangle$



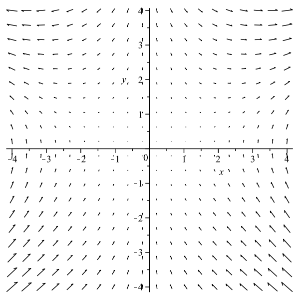
$F(x,y) = \langle y, -x \rangle$



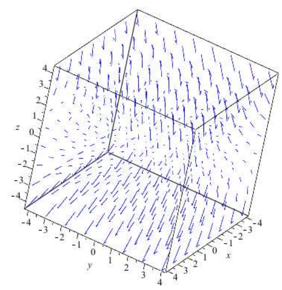
$F(x,y) = \langle y^2, x-1 \rangle$



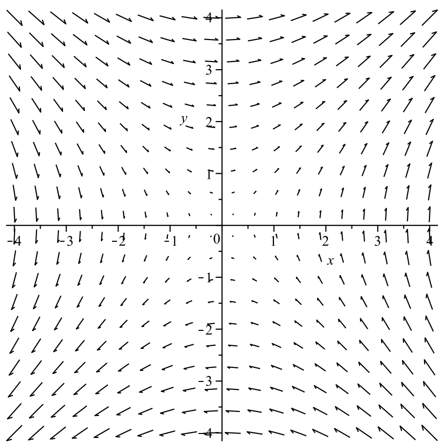
$F(x,y) = \langle 4xy, 2x^2 - 6y \rangle$



$F(x,y,z) = \langle x+y, y+z, -z^2 \rangle$



• A **flow line** through a vector field is the path that a particle would trace out if it were placed in a vector field and acted on by the forces generated by the field. The particle is "pushed" in the direction indicated by the vector its starting point, and onward from there.
Example: Sketch the flow lines through the following the following vector from the initial points: $(2,1)$, $(-1,-1)$, and $(-3,1)$.



Definitions:

• Given a function $f(x,y)$ or $f(x,y,z)$, one can define a vector field $F = \nabla f$. This vector field is called the **gradient field** for the function f.

• We call f the **potential function** for the vector field $F = \nabla f$.

• Whenever a vector field F can be thought of as the gradient field fro some potential function f (i.e. when $F = \nabla f$ for some f), then we say that F is a **conservative vector field**.

Examples:

1. Given $f(x,y) = 2x^2y - 3y^2$ and $g(x,y,z) = xy^2z^3$, find the gradient fields $F = \nabla f$ and $G = \nabla g$

   $F = \langle f_x, f_y \rangle = \langle 4xy, 2x^2 - 6y \rangle$, and $G = \langle g_x, g_y, g_z \rangle = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$

2. Given that $F = \langle 3x^2 - 4xy, -2x^2 \rangle$, determine whether or not F is conservative.

   Suppose that there is a function f such that $f_x = 3x^2 - 4xy$ and $f_y = -2x^2$. Then $f(x,y) = x^3 - 2x^2y + g(y)$ where $g(y)$ is some unknown function of y.

   Differentiating $f(x,y) = x^3 - 2x^2y + g(y)$ with respect to y gives $f_y = -2x^2 + g'(y) = -2x^2$. Therefore, $g'(y) = 0$, and so $f(x,y) = x^3 - 2x^2y + C$. Since $F = \nabla f$ (recheck by computing f_x and f_y), this vector field is conservative.

3. Let $F(x,y,z) = \langle y^2z^2 - 1, 2xyz^2, 2xy^2z + z^3 \rangle$. Determine whether or not F is conservative.

   Again suppose that $F = \nabla f$. Then $f_x = y^2z^2 - 1$, so $f(x,y,z) = xy^2z^2 - x + g(y,z)$. Differentiating this with respect to y gives $f_y = 2xyz^2 + g_y = 2xyz^2$, so $g_y = 0$. Therefore, $g(y,z) = h(z)$, so $f(x,y,z) = xy^2z^2 - x + h(z)$. Diferentiating this with respect to z gives: $f_z = 2xy^2z + h'(z) = 2xy^2z + z^3$. Thus $h'(z) = z^3$. Hence $h(z) = \frac{1}{4}z^4 + C$.

   Therefore, $f(x,y,z) = xy^2z^2 - x + \frac{1}{4}z^4 + C$ is a potential function for F. This F is a conservative vector field.

Finding Equations for Flow Lines: Given a 2D vector field, $F(x,y) = \langle f_1(x,y), f_2(x,y) \rangle$, suppose $\langle x(t), y(t) \rangle$ is the related position function. Then $x'(t) = f_1(x,y)$ and $y'(t) = f_2(x,y)$. Thus, by the chain rule, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{f_2(x,y)}{f_1(x,y)}$.

Example: Given $F(x,y) = \langle y, x \rangle$, we rewrite this using the chain rule (As described above): $\frac{dy}{dx} = \frac{x}{y}$.

Then, using separation of variables, $\int y\,dy = \int x\,dx$, so $\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$. Hence $y^2 = x^2 + 2C$, or $y^2 - x^2 = C'$.

If $C' = 0$, we have $y^2 = x^2$ or $y = \pm x$. Otherwise, we get a hyperbola opening Left/Right when $C' < 0$ and Up/Down when $C' > 0$. [See the vector field graph on the previous page]

Curl and Divergence: Let $F(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z) \rangle$ be a vector field in which M, N, and P all have partial derivatives defined on some region. Then we have the following definitions:

• $curl F = \nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$

• $div F = \nabla \cdot F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$.

Example: Given $F(x,y,z) = \langle x^2y, yx^2, z\sin(xy) \rangle$,

• $curl F = \nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & yx^2 & z\sin(xy) \end{vmatrix}$

$= \left[\frac{\partial}{\partial y}(z\sin(xy)) - \frac{\partial}{\partial z}(yx^2)\right]\vec{i} - \left[\frac{\partial}{\partial x}(z\sin(xy)) - \frac{\partial}{\partial z}(x^2y)\right]\vec{j} + \left[\frac{\partial}{\partial x}(yx^2) - \frac{\partial}{\partial y}(x^2y)\right]\vec{k}$

$= \langle xz\cos(xy), -yz\cos(xy), 2xy - x^2 \rangle$.

• $div F = \nabla \cdot F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = 2xy + x^2 + \sin(xy)$.